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Self-Dual Supergravity and Twistor Theory

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Abstract

By generalizing and extending some of the earlier results derived by Manin and Merkulov, a twistor description is given of four-dimensional $\mathcal{N}$-extended (gauged) self-dual supergravity with and without cosmological constant. Starting from the category of $(4|4\mathcal{N})$-dimensional complex superconformal supermanifolds, the categories of $(4|2\mathcal{N})$-dimensional complex quaternionic, quaternionic Kähler and hyper-Kähler right-chiral supermanifolds are introduced and discussed. We then present a detailed twistor description of these types of supermanifolds. In particular, we construct supertwistor spaces associated with complex quaternionic right-chiral supermanifolds, and explain what additional supertwistor data allows for giving those supermanifolds a hyper-Kähler structure. In this way, we obtain a supersymmetric generalization of Penrose's nonlinear graviton construction. We furthermore give an alternative formulation in terms of a supersymmetric extension of LeBrun's Einstein bundle. This allows us to include the cases with nonvanishing cosmological constant. We also discuss the bundle of local supertwistors and address certain implications thereof. Finally, we comment on a real version of the theory related to Euclidean signature.

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1. Introduction and results

Since the discovery of twistor string theories by Witten [1] and Berkovits [2] about three years ago, a lot of advancements in our understanding of the properties of (supersymmetric) Yang-Mills theory has been made. Despite the fact that these string theories describe supersymmetric Yang-Mills theory coupled to conformal supergravity [3], they provide an elegant way of describing some of the remarkable features exhibited by the scattering amplitudes of the gauge theory (see e.g. [4, 5] and references therein). Surely, the appearance of conformal supergravity is awkward since it is inextricably mixed in with the gauge theory beyond tree-level in perturbation theory. This makes it impossible to solely compute gauge theory scattering amplitudes beyond tree-level by performing a string theory calculation. In addition, one rather wishes to describe Einstein supergravity than conformal supergravity as the latter is believed not to be a suitable candidate for describing nature due to its lack of unitarity. In view of that, Abou-Zeid et al. [6] proposed new twistor string theories which indeed seem to yield supersymmetric Yang-Mills theory coupled to Einstein supergravity. Among the already mentioned aspects, a variety of other related issues has been investigated and is still being explored [7]–[19] (for recent reviews, see also Refs. [20]).

Despite the success, a consistent twistor string formulation of gravity remains an open question. In order to find such a formulation, it is certainly necessary to first understand better the twistor description of Einstein supergravity theories. Before trying to attempt to solve this task in full generality, one may first consider a simplification of the theory by restricting the focus to the much simpler theory of self-dual supergravity. In view of that, recall from the early work by Penrose [21] that it is possible to associate with any complex-Riemannian four-dimensional manifold M (complex space-time) which is equipped with a conformal structure and has self-dual Weyl curvature, a complex three-dimensional twistor space P which is defined to be the space of maximal isotropic (totally null) complex submanifolds of M. All the information about the conformal structure of M is encoded in the complex structure of the twistor space P. Some additional data on P then allows for the construction of self-dual metrics and conformal structures on M. For explicit constructions, see Refs. [22]–[37], for instance. Moreover, hidden symmetries and hierarchies of self-dual gravity have been studied by the authors of [38]–[42]. Notice also that one may return to the realm of Riemannian geometry by restricting the objects under consideration to the fixed-point set of an anti-holomorphic involution.

Self-dual supergravity theories on four-dimensional space-time have first appeared in the works [43]–[47] and have subsequently been discussed, e.g. by the authors of [48, 49] within the harmonic superspace framework (see also Galperin et al. [50] and references therein). The purpose of this article is to give the twistor description of $\mathcal{N}$-extended self-dual supergravity with and without cosmological constant. In particular, we shall generalize and extend the earlier results by Manin [51] and Merkulov [52]–[55].[1] For most of the time, we work in the context of complex supermanifolds but at the end we also discuss a real version of the theory.

[1] Notice that Merkulov [54] has given a twistor description of minimal Einstein supergravity.

In the next section, starting from the category of complex superconformal supermanifolds of dimension $(4|4\mathcal{N})$, the categories of

(i) complex quaternionic right-chiral (hereafter RC) supermanifolds,
(ii) complex quaternionic Kähler RC supermanifolds and
(iii) complex hyper-Kähler RC supermanifolds

are introduced and discussed. In this section, special attention is paid to the construction of the connections and their properties under superconformal rescalings. In Sec. 3., we first discuss the twistor theory of complex quaternionic RC supermanifolds. We shall establish a double fibration of the form

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathscr{P} & & \mathscr{M} \end{tikzcd}$$

where $\mathscr{M}$ is a complex quaternionic RC supermanifold subject to additional restrictions and $\mathscr{P}$ its associated supertwistor space. The supermanifold $\mathscr{F}$ is a certain $\mathbb{P}^1$-bundle over $\mathscr{M}$ and termed correspondence space. In this way, $\mathscr{M}$ is viewed as the space of complex submanifolds of $\mathscr{P}$ which are biholomorphically equivalent to the complex projective line $\mathbb{P}^1$ and have normal sheaf described by

$$0 \longrightarrow \Pi\mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^{\mathcal{N}} \longrightarrow \mathscr{N}_{\mathbb{P}^1|\mathscr{P}} \longrightarrow \mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^2 \longrightarrow 0.$$

Here, Π is the Graßmann parity changing functor and $\mathscr{O}_{\mathbb{P}^1}(1)$ is the sheaf of sections of the dual tautological ($c_1 = 1$) bundle over $\mathbb{P}^1$.

Having established this correspondence, we focus on the twistor description of complex hyper-Kähler RC supermanifolds – the case of interest in view of studying self-dual supergravity with zero cosmological constant. In particular, we give the supersymmetric analog of Penrose's nonlinear graviton construction [21], i.e. we shall show that in this case the supertwistor space is holomorphically fibred over the Riemann sphere $\mathscr{P} \to \mathbb{P}^1$ and equipped with a certain relative symplectic structure. Furthermore, we present an equivalent formulation of the Penrose construction in terms of a supersymmetric generalization of LeBrun's Einstein bundle [56]. This construction allows for including a cosmological constant to the self-dual supergravity equations. In particular, the Einstein bundle is defined over the supertwistor space and as we shall see, its nonvanishing sections are in one-to-one correspondence with solutions to the self-dual supergravity equations with cosmological constant. Requiring its sections to be integrable amounts to putting the cosmological constant to zero. As in the purely bosonic situation, this bundle can explicitly be described in terms of certain intrinsic holomorphic data on the supertwistor space. Besides this, also in Sec. 3., we introduce the bundle of local supertwistors over $\mathscr{M}$ and discuss certain implications thereof. For instance, we shall show that it can be reinterpreted in terms of a certain jet-bundle over the supertwistor space by means of the Penrose-Ward transform. All these considerations are first generic in the sense of keeping arbitrary the number $\mathcal{N}$ of allowed supersymmetries. However, like in the flat situation (see e.g. Witten [1]), the $\mathcal{N} = 4$ case is special and deserves a separate treatment.

Finally, in Sec. 4. we discuss a real version of the theory, that is, we introduce certain real structures (anti-holomorphic involutions) on all the manifolds appearing in the above double fibration such that the underlying (ordinary) manifold of $\mathscr{M}$ is of Euclidean signature. A particular feature of Euclidean signature is that the number of allowed supersymmetries is restricted to be even.

Remarks

Some general remarks are in order. A *complex supermanifold* of dimension $m|n$ is meant to be a ringed space $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$, where $\mathscr{M}$ is a topological space and $\mathscr{O}_{\mathscr{M}}$ is a sheaf of supercommutative ($\mathbb{Z}_2$-graded) rings on $\mathscr{M}$ such that, if we let $\mathscr{N}$ be the ideal subsheaf in $\mathscr{O}_{\mathscr{M}}$ of all nilpotent elements, the following is fulfilled:

(i) $\mathscr{M}_{\text{red}} := (\mathscr{M}, \mathscr{O}_{\text{red}} := \mathscr{O}_{\mathscr{M}}/\mathscr{N})$ is a complex manifold of (complex) dimension m and
(ii) for any point $x \in \mathscr{M}$ there is neighborhood $\mathscr{U} \ni x$ such that $\mathscr{O}_{\mathscr{M}}|_{\mathscr{U}} \cong \mathscr{O}_{\text{red}}(\Lambda^\bullet \mathscr{E})|_{\mathscr{U}}$,

where $\mathscr{E} := \mathscr{N}/\mathscr{N}^2$ is a rank-n locally free sheaf of $\mathscr{O}_{\text{red}}$-modules on $\mathscr{M}$ and $\Lambda^\bullet$ denotes the exterior algebra. We call $\mathscr{E}$ the *characteristic sheaf* of the complex supermanifold $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ and $\mathscr{O}_{\mathscr{M}}$ its *structure sheaf.* See, e.g. Manin [51] for more details. Such supermanifolds are said to be *locally split.* In this work, we will assume that the Graßmann odd directions have trivial topology, that is, we work in the category of *globally split* supermanifolds $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ with $\mathscr{O}_{\mathscr{M}} \cong \mathscr{O}_{\text{red}}(\Lambda^\bullet \mathscr{E})$. For the sake of brevity, we shall be referring to them as split supermanifolds, in the sequel. The structure sheaf $\mathscr{O}_{\mathscr{M}}$ of a split supermanifold admits the $\mathbb{Z}$-grading $\mathscr{O}_{\mathscr{M}} \cong \bigoplus_{p \geq 0} \mathscr{O}^p_{\mathscr{M}}$, where $\mathscr{O}^p_{\mathscr{M}} \cong \mathscr{O}_{\text{red}}(\Lambda^p \mathscr{E})$. Moreover, the assumption of being split implies that there will always exist an atlas $\{\{\mathscr{U}_a\}, \{\varphi_{ab}, \vartheta_{ab}\}\}$ on $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ such that, if we let $(z_a) = (z_a^1, \ldots, z_a^m)$ be Graßmann even coordinates and $(\eta_a) = (\eta_a^1, \ldots, \eta_a^n)$ be Graßmann odd coordinates on the patch $\mathscr{U}_a \subset \mathscr{M}$, the transition functions on nonempty intersections $\mathscr{U}_a \cap \mathscr{U}_b$ are of the form $z_a = \varphi_{ab}(z_b)$ and $\eta_a = (\vartheta^i_{j\,ab}(z_b)\eta_b^j)$ for $i, j = 1, \ldots, n$. We will frequently be working with such atlases without particularly referring to them.

An important example of a split supermanifold is the complex projective superspace $\mathbb{P}^{m|n}$ given by

$$\mathbb{P}^{m|n} \;=\; (\mathbb{P}^m, \mathscr{O}_{\text{red}}(\Lambda^\bullet(\mathscr{O}_{\mathbb{P}^m}(-1) \otimes \mathbb{C}^n))),$$

where $\mathscr{O}_{\mathbb{P}^m}(-1)$ is the sheaf of sections of the tautological ($c_1 = -1$) line bundle over the complex projective space $\mathbb{P}^m$. The reason for the appearance of $\mathscr{O}_{\mathbb{P}^m}(-1)$ is as follows. If we let $(z^0, \ldots, z^m, \eta^1, \ldots, \eta^n)$ be homogeneous coordinates[2] on $\mathbb{P}^{m|n}$, a holomorphic function f on $\mathbb{P}^{m|n}$ has the expansion

$$f \;=\; \sum f_{i_1 \cdots i_r}(z^0, \ldots, z^m)\, \eta^{i_1} \cdots \eta^{i_r}.$$

Surely, for f to be well-defined the total homogeneity of f must be zero. Hence, $f_{i_1 \cdots i_r}$ must be of homogeneity $-r$. This explains the above form of the structure sheaf of the complex

[2] Recall that they are subject to the identification $(z^0, \ldots, z^m, \eta^1, \ldots, \eta^n) \sim (tz^0, \ldots, tz^m, t\eta^1, \ldots, t\eta^n)$, where $t \in \mathbb{C} \setminus \{0\}$.

projective superspace. Of particular interest in the context of (flat) twistor theory is $\mathbb{P}^{3|\mathcal{N}}$, which is the *supertwistor space* associated with the superconformal compactification of the chiral complex superspace $\mathbb{C}^{4|2\mathcal{N}}$.

Moreover, two things are worth mentioning. First, *any* given complex supermanifold is actually a deformation of a split supermanifold (Rothstein [57]), that is, to any complex supermanifold $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ there is associated a complex analytic one-parameter family of complex supermanifolds $(\mathscr{M}, \mathscr{O}_{\mathscr{M},t})$, for $t \in \mathbb{C}$, such that $\mathscr{O}_{\mathscr{M},t=0} \cong \mathscr{O}_{\text{red}}(\Lambda^\bullet \mathscr{E})$ and $\mathscr{O}_{\mathscr{M},t=1} \cong \mathscr{O}_{\mathscr{M}}$, where $\mathscr{E}$ is the characteristic sheaf of $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$. Second, smooth supermanifolds are *always* split due to Batchelor's theorem [58]. The latter result follows because of the existence of a smooth partition of unity in the category of smooth supermanifolds. Since we are eventually interested in self-dual supergravity on a four-dimensional Riemannian manifold, this explains why we may restrict our discussion to split supermanifolds as stated above.

Furthermore, when there is no confusion with the underlying topological space, we denote the supermanifold $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ simply by $\mathscr{M}$. Finally, we point out that (holomorphic) vector bundles $\mathscr{E}$ of rank $r|s$ over some complex supermanifold $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ are meant to be locally free sheaves of $\mathscr{O}_{\mathscr{M}}$-modules, that is, they are locally of the form $\mathscr{O}_{\mathscr{M}} \otimes \mathbb{C}^r \oplus \Pi\mathscr{O}_{\mathscr{M}} \otimes \mathbb{C}^s$. Hence, the notions "vector bundle" and "locally free sheaf" are used interchangeably. This will also allow us to simplify notation. In addition, the dual of any locally free sheaf $\mathscr{E}$ on $\mathscr{M}$ is denoted by

$$\mathscr{E}^\vee \;=\; \mathscr{H}om_{\mathscr{O}_{\mathscr{M}}}(\mathscr{E}, \mathscr{O}_{\mathscr{M}}).$$

For line bundles $\mathscr{L}$, we instead write $\mathscr{L}^{-1}$. If there is no confusion, the dimensionality (respectively, the rank) of ordinary manifolds (respectively, of ordinary vector bundles) will often be abbreviated by $m|0 \equiv m$ (respectively, by $r|0 \equiv r$).

2. Self-dual supergravity

2.1. Superconformal structures

Remember that a (holomorphic) *conformal structure* on an ordinary four-dimensional complex spin manifold M can be introduced in two equivalent ways. The first definition states that a conformal structure is an equivalence class $[g]$, the *conformal class*, of holomorphic metrics g on M, where two given metrics g and g' are called equivalent if $g' = \gamma^2 g$ for some nowhere vanishing holomorphic function γ. Putting it differently, a conformal structure is a line subbundle L in $\Omega^1 M \odot \Omega^1 M$. The second definition assumes a factorization of the holomorphic tangent bundle TM of M as a tensor product of two rank-2 holomorphic vector bundles S and $\tilde{S}$, that is, $TM \cong S \otimes \tilde{S}$. This isomorphism in turn gives (canonically) the line subbundle $\Lambda^2 S^\vee \otimes \Lambda^2 \tilde{S}^\vee$ in $\Omega^1 M \odot \Omega^1 M$ which, in fact, can be identified with L.

Next one needs to extend the notion of a conformal structure to supermanifolds. We shall see that the generalization of the latter of the two approaches given above seems to be the appropriate one for our present purposes (see §2.3. for some remarks regarding standard supergravities). Our subsequent discussion closely follows the one given by Manin [51] and Merkulov [52], respectively.

§2.1. Superconformal supermanifolds. To give the definition of a *superconformal structure*, let $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ be a $(4|4\mathcal{N})$-dimensional complex supermanifold, where $\mathcal{N}$ is a nonnegative integer.

Definition 2.1. *A superconformal structure on $\mathscr{M}$ is a pair of integrable rank-$0|2\mathcal{N}$ distributions $T_l\mathscr{M}$ and $T_r\mathscr{M}$ which obey the following conditions:*

(i) their sum in the holomorphic tangent bundle $T\mathscr{M}$ is direct,

(ii) there exist two rank-$2|0$ locally free sheaves $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ and one rank-$0|\mathcal{N}$ locally free sheaf $\mathscr{E}$ such that $T_l\mathscr{M} \cong \mathscr{S} \otimes \mathscr{E}^\vee$ and $T_r\mathscr{M} \cong \mathscr{E} \otimes \widetilde{\mathscr{S}}$,[3]

(iii) the Frobenius form

$$\begin{aligned} \Phi \,:\, T_l\mathscr{M} \otimes T_r\mathscr{M} \;&\rightarrow\; T_0\mathscr{M} \;:=\; T\mathscr{M}/(T_l\mathscr{M} \oplus T_r\mathscr{M}), \\ (X \otimes Y) \;&\mapsto\; [X, Y\} \bmod (T_l\mathscr{M} \oplus T_r\mathscr{M}) \end{aligned}$$

coincides with the natural map $\mathscr{S} \otimes \mathscr{E}^\vee \otimes \mathscr{E} \otimes \widetilde{\mathscr{S}} \rightarrow \mathscr{S} \otimes \widetilde{\mathscr{S}}$ and gives an isomorphism $T_0\mathscr{M} \cong \mathscr{S} \otimes \widetilde{\mathscr{S}}$. Here, $[\cdot,\cdot\}$ denotes the graded Lie bracket.

From this definition it follows that $T_{l,r}\mathscr{M}$ define two foliations on $\mathscr{M}$. Let us denote the resulting quotients by $\mathscr{M}_{r,l}$. Furthermore, $\mathscr{M}_l$ and $\mathscr{M}_r$ are supermanifolds which are both of dimension $(4|2\mathcal{N})$. In addition, their structure sheaves $\mathscr{O}_{\mathscr{M}_{l,r}}$ are those subsheaves of $\mathscr{O}_{\mathscr{M}}$ which are annihilated by vector fields from $T_{r,l}\mathscr{M}$. By virtue of the inclusions $\mathscr{O}_{\mathscr{M}_{l,r}} \subset \mathscr{O}_{\mathscr{M}}$, we find the following double fibration:

$$\begin{tikzcd} & \mathscr{M} \arrow[dl, "\pi_l"'] \arrow[dr, "\pi_r"] & \\ \mathscr{M}_l & & \mathscr{M}_r \end{tikzcd} \tag{2.1}$$

According to Manin [51], we shall call $\mathscr{M}_l$ and $\mathscr{M}_r$ *left-* and *right-chiral* supermanifolds, respectively. Moreover, we have

$$0 \;\longrightarrow\; T_{l,r}\mathscr{M} \;\longrightarrow\; T\mathscr{M} \;\longrightarrow\; \pi^*_{r,l}T\mathscr{M}_{r,l} \;\longrightarrow\; 0, \tag{2.2}$$

which is induced by the double fibration (2.1). Putting it differently, $T_l\mathscr{M}$ (respectively, $T_r\mathscr{M}$) is the relative tangent sheaf of the fibration $\pi_r : \mathscr{M} \rightarrow \mathscr{M}_r$ (respectively, of $\pi_l : \mathscr{M} \rightarrow \mathscr{M}_l$). Note that a superconformal structure is, by no means, just given by a conformal class of supermetrics.

§2.2. Some properties and an example. First of all, it should be noticed that on the underlying four-dimensional manifold $\mathscr{M}_{\rm red}$, we naturally have the rank-2 holomorphic vector bundles $\mathscr{S}_{\rm red}$ and $\widetilde{\mathscr{S}}_{\rm red}$. Part (iii) of Def. 2.1. then guarantees a factorization of the holomorphic tangent bundle $T\mathscr{M}_{\rm red}$ of $\mathscr{M}_{\rm red}$ as $T\mathscr{M}_{\rm red} \cong \mathscr{S}_{\rm red} \otimes \widetilde{\mathscr{S}}_{\rm red}$. Hence, $\mathscr{M}_{\rm red}$ comes naturally equipped with a conformal structure.

[3] Do not confuse $\mathscr{E}$ with the characteristic sheaf of $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$.

Furthermore, Def. 2.1. implies that the holomorphic tangent bundle $T\mathscr{M}$ of $\mathscr{M}$ fits into the following short exact sequence:

$$0 \longrightarrow T_l\mathscr{M} \oplus T_r\mathscr{M} \stackrel{i_l \oplus i_r}{\longrightarrow} T\mathscr{M} \stackrel{\hat{\Phi}^{-1}}{\longrightarrow} T_0\mathscr{M} \longrightarrow 0, \tag{2.3}$$

where $i_{l,r}$ are the natural inclusion mappings and $\hat{\Phi}$ is the contracted Frobenius form which is invertible by assumption. Recall that $T_0\mathscr{M} \cong \mathscr{S} \otimes \widetilde{\mathscr{S}}$. Consider now the subsheaves $\pi^*_{l,r}T\mathscr{M}_{l,r} \subset T\mathscr{M}$. It can be shown (cf. Manin [51] and Merkulov [52]) that their structure is also described by a short exact sequence similar to the one given above, i.e.

$$0 \longrightarrow T_{l,r}\mathscr{M} \longrightarrow \pi^*_{l,r}T\mathscr{M}_{l,r} \longrightarrow T_0\mathscr{M} \longrightarrow 0. \tag{2.4}$$

This implies that also the underlying manifolds $\mathscr{M}_{l,r\,\mathrm{red}}$ of $\mathscr{M}_{l,r}$ are naturally equipped with conformal structures in the usual sense.

The prime example of the above construction is the flag supermanifold

$$\mathscr{M} = F_{2|0,2|\mathcal{N}}(\mathbb{C}^{4|\mathcal{N}}) = \{S^{2|0} \subset S^{2|\mathcal{N}} \subset \mathbb{C}^{4|\mathcal{N}}\}. \tag{2.5}$$

In this case, the double fibration (2.1) takes the following form:

$$\begin{tikzcd}[column sep=small] & \mathscr{M} = F_{2|0,2|\mathcal{N}}(\mathbb{C}^{4|\mathcal{N}}) \arrow[dl, "\pi_l"'] \arrow[dr, "\pi_r"] & \\ \mathscr{M}_l = F_{2|\mathcal{N}}(\mathbb{C}^{4|\mathcal{N}}) & & \mathscr{M}_r = F_{2|0}(\mathbb{C}^{4|\mathcal{N}}) \end{tikzcd} \tag{2.6}$$

In addition, there are four natural sheaves $\mathscr{S}^{2|0} \subset \mathscr{S}^{2|\mathcal{N}}$ and $\widetilde{\mathscr{S}}^{2|0} \subset \widetilde{\mathscr{S}}^{2|\mathcal{N}}$ on $\mathscr{M}$, where $\mathscr{S}^{2|0}$ and $\mathscr{S}^{2|\mathcal{N}}$ are the two tautological sheaves while the other two are defined by two short exact sequences

$$\begin{aligned} 0 &\longrightarrow \mathscr{S}^{2|0} \longrightarrow \mathscr{O}_{\mathscr{M}} \otimes \mathbb{C}^{4|\mathcal{N}} \longrightarrow (\widetilde{\mathscr{S}}^{2|\mathcal{N}})^\vee \longrightarrow 0, \\ 0 &\longrightarrow \mathscr{S}^{2|\mathcal{N}} \longrightarrow \mathscr{O}_{\mathscr{M}} \otimes \mathbb{C}^{4|\mathcal{N}} \longrightarrow (\widetilde{\mathscr{S}}^{2|0})^\vee \longrightarrow 0. \end{aligned} \tag{2.7}$$

A short calculation shows that these two sequences together with (2.4) imply

$$\mathscr{S} \cong (\mathscr{S}^{2|0})^\vee, \qquad \widetilde{\mathscr{S}} \cong (\widetilde{\mathscr{S}}^{2|0})^\vee \qquad \text{and} \qquad \mathscr{E} \cong \widetilde{\mathscr{S}}^{2|\mathcal{N}}/\widetilde{\mathscr{S}}^{2|0}. \tag{2.8}$$

In addition, one may also verify that points (i) and (iii) of Def. 2.1. are satisfied. For more details, see Manin [51]. The three flag supermanifolds $F_{2|0,2|\mathcal{N}}(\mathbb{C}^{4|\mathcal{N}})$, $F_{2|\mathcal{N}}(\mathbb{C}^{4|\mathcal{N}})$ and $F_{2|0}(\mathbb{C}^{4|\mathcal{N}})$ play an important role in the twistor description of supersymmetric Yang-Mills theories, as they represent the superconformal compactifications of the flat complex superspaces $\mathbb{C}^{4|4\mathcal{N}}$ and $\mathbb{C}^{4|2\mathcal{N}}_{l,r}$ (for recent reviews, see Refs. [20]).

§2.3. Remarks. In this work, we shall only be concerned with $\mathcal{N}$-extended self-dual supergravities. These theories are conveniently formulated on right-chiral supermanifolds, as has been discussed in, e.g., Refs. [47]–[49]. By a slight abuse of notation, we denote $\mathscr{M}_r$ simply by $\mathscr{M}$. Henceforth, we shall be working with (complex) right-chiral (hereafter RC) supermanifolds of dimension $(4|2\mathcal{N})$. Furthermore, for various technical reasons but also for reasons

related to the real structures discussed in Sec. 4., we restrict the number $\mathcal{N}$ of allowed supersymmetries to be *even*. Note that for the complex situation (or the case of split signature), this restriction on $\mathcal{N}$ is not necessary.

Furthermore, let us emphasize that for making contact with standard *full* non-self-dual supergravities with $\mathcal{N} > 2$ (this includes both Einstein and conformal supergravities), a superconformal structure should be defined slightly differently since due to certain torsion constraints (see e.g. Howe [59]), there are no (anti-)chiral superfields. This means that the distributions $T_{l,r}\mathscr{M}$ should be taken to be suitably non-integrable, which means that one does not have the double fibration (2.1). Also for $\mathcal{N} > 2$, one should relax the condition of integrability of $T_l\mathscr{M}$ if one wishes to talk about self-dual supergravity since in that case only $T_r\mathscr{M}$ should be considered to be integrable. For our discussions given below, these issues can be left aside. They certainly deserve further studies in view of complementing the self-dual theory to the full theory.

2.2. Geometry of right-chiral supermanifolds

This section is devoted to some geometric aspects of RC supermanifolds. In particular, we discuss their structure group, introduce scales and vielbeins and talk about connections, torsion and curvature.

§2.4. Structure group. Let $\mathscr{M}$ be an RC supermanifold. From (2.4) we know that the holomorphic tangent bundle $T\mathscr{M}$ of $\mathscr{M}$ is described by a sequence of the form

$$0 \longrightarrow \mathscr{E} \otimes \widetilde{\mathscr{S}} \longrightarrow T\mathscr{M} \longrightarrow \mathscr{S} \otimes \widetilde{\mathscr{S}} \longrightarrow 0, \tag{2.9}$$

where $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ are both of rank $2|0$ and $\mathscr{E}$ is of rank $0|\mathcal{N}$, respectively. By a slight abuse of notation, we are again using the same symbols $\mathscr{S}$, $\widetilde{\mathscr{S}}$ and $\mathscr{E}$. Next we notice that $T\mathscr{M}$ is given by the tensor product $\mathscr{H} \otimes \widetilde{\mathscr{S}}$, where $\mathscr{H} \to \mathscr{M}$ is a rank-$2|\mathcal{N}$ holomorphic vector bundle over $\mathscr{M}$ described by

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{H} \longrightarrow \mathscr{S} \longrightarrow 0. \tag{2.10}$$

Hence, the structure group of $T\mathscr{M}$ is as follows: the supergroup $GL(2|\mathcal{N}, \mathbb{C})$ acts on the left on $\mathscr{H}$ and $GL(2|0, \mathbb{C})$ acts on the right on $\widetilde{\mathscr{S}}$ by inverses. The resulting induced action on the tangent bundle $T\mathscr{M}$ yields a subsupergroup of $GL(4|2\mathcal{N}, \mathbb{C})$. Let us denote it by G and its Lie superalgebra by $\mathfrak{g}$. In addition, there is a $|4-\mathcal{N}|$-fold cover of $G \subset GL(4|2\mathcal{N}, \mathbb{C})$

$$1 \longrightarrow \mathbb{Z}_{|4-\mathcal{N}|} \longrightarrow S(GL(2|\mathcal{N}, \mathbb{C}) \times GL(2|0, \mathbb{C})) \longrightarrow G \longrightarrow 1, \tag{2.11}$$

where $S(GL(2|\mathcal{N}, \mathbb{C}) \times GL(2|0, \mathbb{C}))$ is the subsupergroup of $SL(4|\mathcal{N}, \mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} A_1 & 0 & A_2 \\ 0 & b & 0 \\ A_3 & 0 & A_4 \end{pmatrix}, \qquad \text{with} \qquad \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in GL(2|\mathcal{N}, \mathbb{C}) \tag{2.12}$$

and $b \in GL(2|0, \mathbb{C})$. Here, the A_is, for $i = 1, \ldots, 4$, are the defining blocks of a supermatrix in standard format, that is, the matrices $A_{1,4}$ are Graßmann even while the $A_{2,3}$ are Graßmann odd (see e.g. Manin [51] for more details).

§2.5. Scales and vielbeins. Let us start by defining what shall be understood by a *scale* on an RC supermanifold $\mathscr{M}$ (cf. also Refs. [51, 52, 60, 55]). Consider the sequences (2.9) and (2.10). They immediately give the following natural isomorphisms of Berezinian sheaves:

$$\operatorname{Ber} T\mathscr{M} \;\cong\; (\operatorname{Ber}\mathscr{S})^2 \otimes (\operatorname{Ber}\mathscr{E})^2 \otimes (\operatorname{Ber}\widetilde{\mathscr{S}})^{2-\mathcal{N}} \;\cong\; (\operatorname{Ber}\mathscr{H})^2 \otimes (\operatorname{Ber}\widetilde{\mathscr{S}})^{2-\mathcal{N}}. \tag{2.13}$$

Hence, the Berezinian sheaf $\operatorname{Ber}(\mathscr{M}) := \operatorname{Ber}\Omega^1\mathscr{M}$ of $\mathscr{M}$ is

$$\operatorname{Ber}(\mathscr{M}) \;\cong\; (\operatorname{Ber}\mathscr{H}^\vee)^2 \otimes (\operatorname{Ber}\widetilde{\mathscr{S}}^\vee)^{2-\mathcal{N}}. \tag{2.14}$$

In addition, we have $\operatorname{Ber}\widetilde{\mathscr{S}} \cong \Lambda^2\widetilde{\mathscr{S}}$ since $\widetilde{\mathscr{S}}$ is of purely even rank $2|0$. Moreover, without loss of generality, one may always make the identification

$$\operatorname{Ber}\mathscr{H} \;\cong\; \operatorname{Ber}\widetilde{\mathscr{S}}. \tag{2.15}$$

In this respect, recall that $\mathcal{N}$ is assumed to be even. In fact, since the tangent bundle can be factorized as $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$, one can locally choose to have such an isomorphism.

Then we may give the following definition:

Definition 2.2. *A scale on an RC supermanifold $\mathscr{M}$ is a choice of a particular non-vanishing volume form $\tilde{\varepsilon} \in H^0(\mathscr{M}, \operatorname{Ber}\widetilde{\mathscr{S}}^\vee)$ on the vector bundle $\widetilde{\mathscr{S}}$. A superconformal rescaling is a change of scale.*

Therefore, together with the identification $\operatorname{Ber}\mathscr{H}^\vee \cong \operatorname{Ber}\widetilde{\mathscr{S}}^\vee$, a section of $\operatorname{Ber}\widetilde{\mathscr{S}}^\vee$ gives a section of $\operatorname{Ber}(\mathscr{M})$ on $\mathscr{M}$. We shall denote a generic section of $\operatorname{Ber}(\mathscr{M})$ by Vol, in the sequel.[4]

It is well known that a particular choice of a coordinate system on any supermanifold determines the corresponding trivialization of the (co)tangent bundle and hence, of the Berezinian bundle. Let now $\mathscr{U}$ be an open subset of $\mathscr{M}$. On $\mathscr{U}$ we may introduce $(x^{\mu\dot\nu}, \eta^{m\dot\mu})$ as local coordinates, where $\mu, \nu, \ldots = 1, 2$, $\dot\mu, \dot\nu, \ldots = \dot 1, \dot 2$ and $m, n, \ldots = 1, \ldots, \mathcal{N}$. The entire set of coordinates is denoted by $x^{\mathbf{M}}$, where $\mathbf{M} = (\mu\dot\nu, m\dot\mu)$ is an Einstein index. We shall also make use of the notation $\mathbf{M} = M\dot\mu$, where $M = (\mu, m)$. Then $\partial/\partial x^{\mathbf{M}}$ (respectively, $\mathrm{d}x^{\mathbf{M}}$) are basis sections of the tangent bundle $T\mathscr{M}$ (respectively, of the cotangent bundle $\Omega^1\mathscr{M}$) of $\mathscr{M}$. We may associate with the set $\{\partial/\partial x^{\mathbf{M}}\}$ (respectively, with $\{\mathrm{d}x^{\mathbf{M}}\}$) a basis section of $\operatorname{Ber}(\mathscr{M})$ which we denote by $D^{-1}(\partial/\partial x^{\mathbf{M}})$ (respectively, by $D(\mathrm{d}x^{\mathbf{M}})$). An arbitrary (local) section of $\operatorname{Ber}(\mathscr{M})$ then takes the following form:

$$\mathrm{Vol} \;=\; \phi\, D^{-1}\left(\frac{\partial}{\partial x^{\mathbf{M}}}\right) \;=\; \phi\, D(\mathrm{d}x^{\mathbf{M}}) \;=:\; \phi\, \mathrm{d}^4x\, \mathrm{d}^{2\mathcal{N}}\eta, \tag{2.16}$$

where ϕ is a nonvanishing function on $\mathscr{U} \subset \mathscr{M}$. In the last step in the above equation, we have introduced a more conventional notation for the volume form.

Next we introduce (local) frame fields $E_{\mathbf{A}}$, which generate the tangent bundle $T\mathscr{M}$, by setting

$$\mathrm{Vol} \;=\; D^{-1}(E_{\mathbf{A}}), \qquad \text{with} \qquad E_{\mathbf{A}} \;:=\; E_{\mathbf{A}}{}^{\mathbf{M}}\frac{\partial}{\partial x^{\mathbf{M}}}. \tag{2.17}$$

[4]Later on, we additionally require that the resulting volume form $\mathrm{Vol} \in H^0(\mathscr{M}, \operatorname{Ber}(\mathscr{M}))$ obeys $\rho(\mathrm{Vol}) = \mathrm{Vol}$, where ρ is a real structure on $\mathscr{M}$. See Sec. 4. for more details.

Obviously, frame fields are unique up to SG-transformations of the form $E_{\mathbf{A}} \mapsto C_{\mathbf{A}}{}^{\mathbf{B}} E_{\mathbf{B}}$, where $C = (C_{\mathbf{A}}{}^{\mathbf{B}})$ is an SG-valued function on $\mathscr{U}$ with SG being the subsupergroup of G described by

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SL(2|\mathcal{N},\mathbb{C}) \times SL(2|0,\mathbb{C}) \longrightarrow SG \longrightarrow 1, \tag{2.18}$$

as $\mathcal{N}$ is assumed to be even. Putting it differently, a choice of scale on $\mathscr{M}$ reduces the structure group from G to SG. By comparing (2.16) with (2.17), we see that the function ϕ is given by the superdeterminant of $E_{\mathbf{A}}{}^{\mathbf{M}}$, i.e. $\phi = \mathrm{Ber}(E_{\mathbf{A}}{}^{\mathbf{M}})$. This particular frame is also called the *structure frame*. In addition, (local) coframe fields $E^{\mathbf{A}}$ are given by

$$E^{\mathbf{A}} := \mathrm{d}x^{\mathbf{M}} E_{\mathbf{M}}{}^{\mathbf{A}}, \qquad \text{with} \qquad E_{\mathbf{A}} \lrcorner E^{\mathbf{B}} = \delta_{\mathbf{A}}{}^{\mathbf{B}}. \tag{2.19}$$

They generate the cotangent bundle $\Omega^1\mathscr{M}$ of $\mathscr{M}$. Here, $E_{\mathbf{A}}{}^{\mathbf{M}}$ and $E_{\mathbf{M}}{}^{\mathbf{A}}$ are called *vielbein matrices* which obey

$$E_{\mathbf{A}}{}^{\mathbf{M}} E_{\mathbf{M}}{}^{\mathbf{B}} = \delta_{\mathbf{A}}{}^{\mathbf{B}} \qquad \text{and} \qquad E_{\mathbf{M}}{}^{\mathbf{A}} E_{\mathbf{A}}{}^{\mathbf{N}} = \delta_{\mathbf{M}}{}^{\mathbf{N}}. \tag{2.20}$$

The *structure group* or *structure frame* indices $\mathbf{A}, \mathbf{B}, \ldots$ look explicitly as $\mathbf{A} = (\alpha\dot\beta, i\dot\alpha)$, where $\alpha, \beta, \ldots = 1, 2$, $\dot\alpha, \dot\beta, \ldots = \dot 1, \dot 2$ and $i, j, \ldots = 1, \ldots, \mathcal{N}$, respectively. Again, we shall write $\mathbf{A} = A\dot\alpha$ with $A = (\alpha, i)$.

Recall that by virtue of (2.14) and (2.15), a section of $\mathrm{Ber}\, \widetilde{\mathscr{S}}^\vee$ gives a section of $\mathrm{Ber}(\mathscr{M})$. If we rescale this section by some nonvanishing function γ, the volume form Vol changes as $\mathrm{Vol} \mapsto \widehat{\mathrm{Vol}} = \gamma^{4-\mathcal{N}}\mathrm{Vol}$. Up to SG-transformations (which can always be reabsorbed in the definition of the vielbein), the frame and coframe fields change accordingly as $E_{\mathbf{A}} \mapsto E_{\widehat{\mathbf{A}}} = \gamma^{-\kappa} E_{\mathbf{A}}$ and $E^{\mathbf{A}} \mapsto E^{\widehat{\mathbf{A}}} = \gamma^{\kappa} E^{\mathbf{A}}$, where

$$\kappa := \frac{4-\mathcal{N}}{4-2\mathcal{N}}. \tag{2.21}$$

§2.6. Connection. Generally speaking, an affine connection ∇ on $\mathscr{M}$ is a Graßmann *even* mapping on the tangent bundle $T\mathscr{M}$,

$$\nabla : T\mathscr{M} \to T\mathscr{M} \otimes \Omega^1\mathscr{M}, \tag{2.22}$$

which satisfies the Leibniz formula

$$\nabla(fX) = \mathrm{d}f \otimes X + f\nabla X, \tag{2.23}$$

where f is a local holomorphic function and X a local section of $T\mathscr{M}$. Setting $\nabla_{\mathbf{A}} := E_{\mathbf{A}} \lrcorner \nabla$, we may write Eq. (2.23) explicitly as

$$\nabla_{\mathbf{A}}(fX) = (E_{\mathbf{A}} f)X + (-)^{p_f p_{\mathbf{A}}} f \nabla_{\mathbf{A}} X. \tag{2.24}$$

Here, $p \in \mathbb{Z}_2$ denotes the Graßmann parity. Since $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$, we have the decomposition

$$\nabla = \nabla_{\mathscr{H}} \otimes \mathrm{id}_{\widetilde{\mathscr{S}}} + \mathrm{id}_{\mathscr{H}} \otimes \nabla_{\widetilde{\mathscr{S}}}, \tag{2.25}$$

where

$$\nabla_{\mathscr{H}} : \mathscr{H} \rightarrow \mathscr{H} \otimes \Omega^1\mathscr{M} \qquad \text{and} \qquad \nabla_{\widetilde{\mathscr{S}}} : \widetilde{\mathscr{S}} \rightarrow \widetilde{\mathscr{S}} \otimes \Omega^1\mathscr{M} \tag{2.26}$$

are the two connections on $\mathscr{H}$ and $\widetilde{\mathscr{S}}$, respectively. Locally, the connection ∇ is given in terms of a $\mathfrak{g}$-valued connection one-form $\Omega = (\Omega_{\mathbf{A}}{}^{\mathbf{B}}) = (E^{\mathbf{C}}\Omega_{\mathbf{CA}}{}^{\mathbf{B}})$ which is defined by

$$\nabla E_{\mathbf{A}} \;=\; \Omega_{\mathbf{A}}{}^{\mathbf{B}} E_{\mathbf{B}}, \tag{2.27}$$

with

$$\Omega_{\mathbf{A}}{}^{\mathbf{B}} \;=\; \Omega_{A\dot\alpha}{}^{B\dot\beta} \;=\; \Omega_A{}^B\delta_{\dot\alpha}{}^{\dot\beta} + \delta_A{}^B\Omega_{\dot\alpha}{}^{\dot\beta}, \tag{2.28}$$

by virtue of (2.25). Therefore, Eqs. (2.27) read explicitly as

$$\nabla E_{\mathbf{A}} \;=\; \nabla E_{A\dot\alpha} \;=\; \Omega_A{}^B E_{B\dot\alpha} + \Omega_{\dot\alpha}{}^{\dot\beta} E_{A\dot\beta}.$$

In the following, we shall not make any notational distinction between the three connections ∇, $\nabla_{\mathscr{H}}$ and $\nabla_{\widetilde{\mathscr{S}}}$ and simply denote them commonly by ∇. It will be clear from the context which of those is actually being considered.

§2.7. Torsion. If we set

$$[E_{\mathbf{A}}, E_{\mathbf{B}}\} \;=\; f_{\mathbf{AB}}{}^{\mathbf{C}} E_{\mathbf{C}}, \tag{2.29}$$

where $f_{\mathbf{AB}}{}^{\mathbf{C}}$ are the structure functions, the components of the torsion $T = T^{\mathbf{A}}E_{\mathbf{A}} = \frac{1}{2}E^{\mathbf{B}} \wedge E^{\mathbf{A}}T_{\mathbf{AB}}{}^{\mathbf{C}}E_{\mathbf{C}}$ of ∇, which is defined by

$$T^{\mathbf{A}} \;=\; -\nabla E^{\mathbf{A}} \;=\; -\mathrm{d}E^{\mathbf{A}} + E^{\mathbf{B}} \wedge \Omega_{\mathbf{B}}{}^{\mathbf{A}}, \tag{2.30}$$

are given by

$$T_{\mathbf{AB}}{}^{\mathbf{C}} \;=\; \Omega_{\mathbf{AB}}{}^{\mathbf{C}} - (-)^{p_{\mathbf{A}}p_{\mathbf{B}}}\Omega_{\mathbf{BA}}{}^{\mathbf{C}} - f_{\mathbf{AB}}{}^{\mathbf{C}}. \tag{2.31}$$

Note that if we consider the space of differential two-forms on $\mathscr{M}$, we have

$$\Lambda^2\Omega^1\mathscr{M} \;\cong\; \Lambda^2(\mathscr{H}^\vee \otimes \widetilde{\mathscr{S}}^\vee) \;\cong\; (\Lambda^2\mathscr{H}^\vee \otimes \odot^2\widetilde{\mathscr{S}}^\vee) \oplus (\odot^2\mathscr{H}^\vee \otimes \Lambda^2\widetilde{\mathscr{S}}^\vee), \tag{2.32}$$

where $\odot^p$ denotes the p-th (graded) symmetric power of the bundles in question. Therefore, T can be decomposed as

$$T \;=\; T^- + T^+, \tag{2.33}$$

with

$$T^- \;\in\; H^0(\mathscr{M}, \Lambda^2\mathscr{H}^\vee \otimes \odot^2\widetilde{\mathscr{S}}^\vee \otimes T\mathscr{M}) \quad \text{and} \quad T^+ \;\in\; H^0(\mathscr{M}, \odot^2\mathscr{H}^\vee \otimes \Lambda^2\widetilde{\mathscr{S}}^\vee \otimes T\mathscr{M}). \tag{2.34}$$

In the structure frame, $T^\mp$ look in components as

$$T^- \;:\; T_{A(\dot\alpha B\dot\beta)}{}^{C\dot\gamma} \qquad \text{and} \qquad T^+ \;:\; T_{A[\dot\alpha B\dot\beta]}{}^{C\dot\gamma}, \tag{2.35}$$

where parentheses denote normalized symmetrization while square brackets denote normalized antisymmetrization, respectively.

A tensor is called *totally trace-free* if *all* possible supertraces with respect to upper and lower indices vanish. Then we have the following proposition:

Proposition 2.1. *On any RC supermanifold $\mathscr{M}$ with fixed scale, the totally trace-free parts of T^- and of T^+ are independent of the choice of connection, i.e. they are invariants of $\mathscr{M}$.*

Proof: Here, we are following ideas of Ref. [60] but adopted to the supersymmetric setting. Recall that $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$. Let μ^A be a section of $\mathscr{H}$ and $\lambda^{\dot\alpha}$ be a section of $\widetilde{\mathscr{S}}$, respectively. For fixed scale, a general change of a given connection $\nabla_{A\dot\alpha} = E_{A\dot\alpha} \lrcorner \nabla$ to another one $\widehat{\nabla}_{A\dot\alpha} = E_{A\dot\alpha} \lrcorner \widehat{\nabla}$ is given in terms of the contorsion tensors $\Theta_{A\dot\alpha B}{}^C$ and $\Theta_{A\dot\alpha\dot\beta}{}^{\dot\gamma}$ by

$$(\widehat{\nabla}_{A\dot\alpha} - \nabla_{A\dot\alpha})\mu^B \;=\; (-)^{p_A p_C}\mu^C \Theta_{A\dot\alpha C}{}^B \qquad \text{and} \qquad (\widehat{\nabla}_{A\dot\alpha} - \nabla_{A\dot\alpha})\lambda^{\dot\beta} \;=\; \lambda^{\dot\gamma}\Theta_{A\dot\alpha\dot\gamma}{}^{\dot\beta}.$$

Hence, for a section $u^{A\dot\alpha}$ of $T\mathscr{M}$ this implies

$$(\widehat{\nabla}_{A\dot\alpha} - \nabla_{A\dot\alpha})u^{B\dot\beta} \;=\; (-)^{p_A p_C} u^{C\dot\gamma}\Theta_{A\dot\alpha C\dot\gamma}{}^{B\dot\beta},$$

with

$$\Theta_{A\dot\alpha B\dot\beta}{}^{C\dot\gamma} \;=\; \Theta_{A\dot\alpha B}{}^C \delta_{\dot\beta}{}^{\dot\gamma} + \delta_B{}^C \Theta_{A\dot\alpha\dot\beta}{}^{\dot\gamma}.$$

Note that $p_{\mathbf{A}} \equiv p_A$. From

$$[\widehat{\nabla}_{A\dot\alpha}, \widehat{\nabla}_{B\dot\beta}\} f \;=\; -\widehat{T}_{A\dot\alpha B\dot\beta}{}^{C\dot\gamma}\widehat{\nabla}_{C\dot\gamma} f,$$

where f is a local section of $\mathscr{O}_{\mathscr{M}}$, and from similar expressions for unhatted quantities, we thus obtain

$$\begin{aligned}
\widehat{T}_{A(\dot\alpha B\dot\beta)}{}^{C\dot\gamma} \;&=\; T_{A(\dot\alpha B\dot\beta)}{}^{C\dot\gamma} - 2\Theta_{[A(\dot\alpha B\}}{}^C \delta_{\dot\beta)}{}^{\dot\gamma} - 2\Theta_{[A(\dot\alpha\dot\beta)}{}^{\dot\gamma}\delta_{B\}}{}^C,\\
\widehat{T}_{A[\dot\alpha B\dot\beta]}{}^{C\dot\gamma} \;&=\; T_{A[\dot\alpha B\dot\beta]}{}^{C\dot\gamma} - 2\Theta_{\{A[\dot\alpha B]}{}^C \delta_{\dot\beta]}{}^{\dot\gamma} - 2\Theta_{\{A[\dot\alpha\dot\beta]}{}^{\dot\gamma}\delta_{B]}{}^C,
\end{aligned} \tag{2.36}$$

where $[\cdot\}$ denotes normalized graded antisymmetrization of the enclosed indices while $\{\cdot]$ means normalized graded symmetrization. These expressions make it obvious that changes in the connection are only reflected in the trace parts of the torsion.

$\square$

Therefore, without loss of generality, we can always work with a connection ∇ on $\mathscr{M}$ whose torsion tensors T^- and T^+ are totally trace-free, since given any two connections on the bundles $\mathscr{H}$ and $\widetilde{\mathscr{S}}$ it is always possible to find contorsion tensors such the resulting connection induced on the tangent bundle $T\mathscr{M}$ will be totally trace-free. Indeed, we have the following proposition:

Proposition 2.2. *On any RC supermanifold $\mathscr{M}$ with fixed scale $\tilde\varepsilon \in H^0(\mathscr{M}, \mathrm{Ber}\,\widetilde{\mathscr{S}}^\vee)$ there always exits a connection such that:*

(i) the torsion tensors T^- and T^+ are totally trace-free and

(ii) in addition we have that

$$\nabla\varepsilon \;=\; 0 \;=\; \nabla\tilde\varepsilon,$$

where $\varepsilon \in H^0(\mathscr{M}, \mathrm{Ber}\,\mathscr{H}^\vee)$ is determined by $\tilde\varepsilon$ via the isomorphism (2.15).

Furthermore, for $\mathcal{N} \neq 4$, this connection is unique.

Proof: Existence is clear from our above discussion. It remains to prove uniqueness for $\mathcal{N} \neq 4$. First of all, one notices that given two connections ∇ and $\widetilde{\nabla}$ whose torsion tensors $T^{\mp}$ and $\widetilde{T}^{\mp}$ are totally trace-free, then their contorsion tensors Θ and $\widetilde{\Theta}$ (obtained from an arbitrary connection one has started with) can only differ by the following terms:

$$\begin{aligned}\widehat{\Theta}_{A\dot\alpha B}{}^{C} &:= \widetilde{\Theta}_{A\dot\alpha B}{}^{C} - \Theta_{A\dot\alpha B}{}^{C} = X_{\{A\dot\alpha}\delta_{B]}{}^{C} + Y_{[A\dot\alpha}\delta_{B\}}{}^{C},\\ \widehat{\Theta}_{A\dot\alpha\dot\beta}{}^{\dot\gamma} &:= \widetilde{\Theta}_{A\dot\alpha\dot\beta}{}^{\dot\gamma} - \Theta_{A\dot\alpha\dot\beta}{}^{\dot\gamma} = -Y_{A(\dot\alpha}\delta_{\dot\beta)}{}^{\dot\gamma} - X_{A[\dot\alpha}\delta_{\dot\beta]}{}^{\dot\gamma},\end{aligned}$$

where $X_{A\dot\alpha}$ and $Y_{A\dot\alpha}$ are arbitrary differential one-forms on $\mathscr{M}$. This can be seen upon inspecting the Eqs. (2.36). Next one picks a volume form $\tilde\varepsilon \in H^0(\mathscr{M}, \mathrm{Ber}\,\widetilde{\mathscr{S}}^\vee)$ and hence a volume form $\varepsilon \in H^0(\mathscr{M}, \mathrm{Ber}\,\mathscr{H}^\vee)$. In a structure frame, they are of the form (recall that $\mathscr{M}$ is split)

$$\varepsilon_{\alpha\beta}{}^{i_1\cdots i_\mathcal{N}} = \epsilon_{\alpha\beta}\epsilon^{i_1\cdots i_\mathcal{N}} \qquad \text{and} \qquad \tilde\varepsilon_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\beta},$$

where the ϵ-tensors are totally antisymmetric with $\epsilon_{12} = \epsilon_{\dot1\dot2} = -\epsilon^{1\cdots\mathcal{N}} = -1$. Since $\mathcal{N}$ is assumed to be even, we find

$$\begin{aligned}\widetilde{\nabla}_{A\dot\alpha}\varepsilon_{\beta\gamma}{}^{j_1\cdots j_\mathcal{N}} &= \nabla_{A\dot\alpha}\varepsilon_{\beta\gamma}{}^{j_1\cdots j_\mathcal{N}} - 2\widehat{\Theta}_{A\dot\alpha[\beta}{}^{\delta}\varepsilon_{\delta\gamma]}{}^{j_1\cdots j_\mathcal{N}} + \mathcal{N}\varepsilon_{\beta\gamma}{}^{[j_1\cdots j_{\mathcal{N}-1}k}\widehat{\Theta}_{A\dot\alpha k}{}^{j_\mathcal{N}]},\\ \widetilde{\nabla}_{A\dot\alpha}\varepsilon_{\dot\beta\dot\gamma} &= \nabla_{A\dot\alpha}\varepsilon_{\dot\beta\dot\gamma} - 2\widehat{\Theta}_{A\dot\alpha[\dot\beta}{}^{\dot\delta}\varepsilon_{\dot\delta\dot\gamma]}.\end{aligned}$$

These equations in turn imply that

$$(-)^B\widehat{\Theta}_{A\dot\alpha B}{}^{B} = \widehat{\Theta}_{A\dot\alpha\beta}{}^{\beta} - \widehat{\Theta}_{A\dot\alpha j}{}^{j} = 0 = \widehat{\Theta}_{A\dot\alpha\dot\beta}{}^{\dot\beta},$$

since both, ∇ and $\widetilde{\nabla}$ are assumed to annihilate ε and $\tilde\varepsilon$, respectively. It is then a rather straightforward exercise to verify that $X_{A\dot\alpha}$ and $Y_{A\dot\alpha}$ must vanish for $\mathcal{N} \neq 4$. Hence, $\widetilde{\nabla} = \nabla$ and the proof is completed.

□

Henceforth, we shall be working with a connection on $\mathscr{M}$ which has totally trace-free torsion tensors $T^{\mp}$. Note that if T^{+} is taken to be totally trace-free, it must vanish identically. This is seen as follows. One first notices that $T_{A[\dot\alpha B\dot\beta]}{}^{C\dot\gamma} = \epsilon_{\dot\alpha\dot\beta}T_{AB}{}^{C\dot\gamma}$ as the rank of $\widetilde{\mathscr{S}}$ is $2|0$. Since T^{-} is totally trace-free, it follows from

$$T_{A[\dot\alpha B\dot\beta]}{}^{C\dot\beta} = \epsilon_{\dot\alpha\dot\beta}T_{AB}{}^{C\dot\beta} = 0$$

that $T_{AB}{}^{C\dot\gamma} = 0$. Altogether, the torsion tensor takes the form

$$T = T^{-}. \tag{2.37}$$

Definition 2.3. *An RC supermanifold $\mathscr{M}$ is said to be complex quaternionic it is equipped with a torsion-free connection which annihilates both volume forms ε and $\tilde\varepsilon$.*

For our later discussions, we need to know how a connection behaves under changes of scale.

Proposition 2.3. *Suppose we are given an RC supermanifold $\mathscr{M}$ which is equipped with a connection ∇ that obeys conditions (i) of (ii) given in Prop. 2.2. Suppose further that $\mathcal{N} \neq 4$. Under a rescaling of the form $\tilde{\varepsilon} \mapsto \gamma\tilde{\varepsilon}$, where γ is a nonvanishing holomorphic function, the change of connection to the new one $\widehat{\nabla}$ is given by the following contorsion tensors:*

$$\begin{aligned} \Theta_{A\dot{\alpha}B}{}^{C} &= (-)^{p_A p_B}\gamma_{B\dot{\alpha}}\delta_A{}^C - \tfrac{\kappa}{2}\gamma_{A\dot{\alpha}}\delta_B{}^C, \\ \Theta_{A\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}} &= \gamma_{A\dot{\beta}}\delta_{\dot{\alpha}}{}^{\dot{\gamma}} - \tfrac{\kappa}{2}\gamma_{A\dot{\alpha}}\delta_{\dot{\beta}}{}^{\dot{\gamma}}. \end{aligned} \tag{2.38}$$

Here, $\gamma_{A\dot{\alpha}} := E_{A\dot{\alpha}} \log\gamma$ and the constant κ has been introduced in (2.21)*. This implies that the new connection $\widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}} = E_{\widehat{A}\widehat{\dot{\alpha}}} \lrcorner \widehat{\nabla}$, with $E_{\widehat{A}\widehat{\dot{\alpha}}} = \gamma^{-\kappa}E_{A\dot{\alpha}}$, acts as follows:*

$$\begin{aligned} \nabla_{A\dot{\alpha}}\mu^B &\mapsto \widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\mu^{\widehat{B}} = \gamma^{-\frac{1}{2}\kappa}(\nabla_{A\dot{\alpha}}\mu^B + \delta_A{}^B\mu^C\gamma_{C\dot{\alpha}}), \\ \nabla_{A\dot{\alpha}}\lambda^{\dot{\beta}} &\mapsto \widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\lambda^{\widehat{\dot{\beta}}} = \gamma^{-\frac{1}{2}\kappa}(\nabla_{A\dot{\alpha}}\lambda^{\dot{\beta}} + \delta_{\dot{\alpha}}{}^{\dot{\beta}}\lambda^{\dot{\gamma}}\gamma_{A\dot{\gamma}}), \\ \nabla_{A\dot{\alpha}}\mu_B &\mapsto \widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\mu_{\widehat{B}} = \gamma^{-\frac{3}{2}\kappa}(\nabla_{A\dot{\alpha}}\mu_B - \mu_A\gamma_{B\dot{\alpha}}), \\ \nabla_{A\dot{\alpha}}\lambda_{\dot{\beta}} &\mapsto \widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\lambda_{\widehat{\dot{\beta}}} = \gamma^{-\frac{3}{2}\kappa}(\nabla_{A\dot{\alpha}}\lambda_{\dot{\beta}} - \gamma_{A\dot{\beta}}\lambda_{\dot{\alpha}}), \end{aligned} \tag{2.39}$$

where μ^A and $\lambda^{\dot{\alpha}}$ are sections of the vector bundles $\mathscr{H}$ and $\widetilde{\mathscr{S}}$, respectively, together with $\mu^{\widehat{A}} = \gamma^{\frac{1}{2}\kappa}\mu^A$ and $\lambda^{\widehat{\dot{\alpha}}} = \gamma^{\frac{1}{2}\kappa}\lambda^{\dot{\alpha}}$ and similarly for their duals.

Proof: The first thing one notices is that the components of the volume forms $\varepsilon_{\alpha\beta}{}^{i_1\cdots i_{\mathcal{N}}}$ and $\tilde{\varepsilon}_{\dot{\alpha}\dot{\beta}}$ scale as

$$\varepsilon_{\alpha\beta}{}^{i_1\cdots i_{\mathcal{N}}} \mapsto \widehat{\varepsilon}_{\widehat{\alpha}\widehat{\beta}}{}^{\widehat{i}_1\cdots\widehat{i}_{\mathcal{N}}} = \gamma^{\frac{\mathcal{N}}{4}}\varepsilon_{\alpha\beta}{}^{i_1\cdots i_{\mathcal{N}}} \qquad \text{and} \qquad \tilde{\varepsilon}_{\dot{\alpha}\dot{\beta}} \mapsto \widehat{\tilde{\varepsilon}}_{\widehat{\dot{\alpha}}\widehat{\dot{\beta}}} = \gamma^{-\frac{\mathcal{N}}{4-2\mathcal{N}}}\tilde{\varepsilon}_{\dot{\alpha}\dot{\beta}}.$$

Hence, the conditions

$$\widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\widehat{\varepsilon}_{\widehat{\alpha}\widehat{\beta}}{}^{\widehat{i}_1\cdots\widehat{i}_{\mathcal{N}}} = 0 = \widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\widehat{\tilde{\varepsilon}}_{\widehat{\dot{\alpha}}\widehat{\dot{\beta}}}$$

yield

$$(-)^B\Theta_{A\dot{\alpha}B}{}^B = \Theta_{A\dot{\alpha}\beta}{}^\beta - \Theta_{A\dot{\alpha}j}{}^j = \tfrac{\mathcal{N}}{4}\gamma_{A\dot{\alpha}} \qquad \text{and} \qquad \Theta_{A\dot{\alpha}\dot{\beta}}{}^{\dot{\beta}} = -\tfrac{\mathcal{N}}{4-2\mathcal{N}}\gamma_{A\dot{\alpha}}.$$

Furthermore, by the requirement that the parts $\widehat{T}^{\mp}$ of $\widehat{T}$ are totally trace-free, we find that (see also the proof of Prop. 2.2.)

$$\Theta_{[A\dot{\alpha}B\dot{\beta}\}}{}^{C\dot{\gamma}} = -\kappa\,\gamma_{[A\dot{\alpha}}\delta_{B\dot{\beta}\}}{}^{C\dot{\gamma}}.$$

Combining these results, we arrive after some algebra at Eqs. (2.38). Finally, Eqs. (2.39) follow upon application of $\widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}$ on the appropriate sections.

□

§2.8. Levi-Civita connection. In the class of affine connections on $\mathscr{M}$ which have totally trace-free torsion tensors $T^{\mp}$ there exists a supersymmetric analog of the *Levi-Civita connection.*

Proposition 2.4. *Let $\mathscr{M}$ be an RC supermanifold. For a given pair of non-degenerate sections $e \in H^0(\mathscr{M}, \Lambda^2\mathscr{H}^\vee)$ and $\tilde{\varepsilon} \in H^0(\mathscr{M}, \Lambda^2\widetilde{\mathscr{S}}^\vee)$ there always exists a unique torsion-free connection D on $\mathscr{M}$ such that*

$$De \;=\; 0 \;=\; D\tilde{\varepsilon}.$$

In addition, there is a unique scale (up to multiplicative constants) for which this connection coincides with the one given by Prop. 2.2.

Proof: One first notices that

$$g \;:=\; e \otimes \tilde{\varepsilon} \in H^0(\mathscr{M}, \Lambda^2\mathscr{H}^\vee \otimes \Lambda^2\widetilde{\mathscr{S}}^\vee) \;\subset\; H^0(\mathscr{M}, \Omega^1\mathscr{M} \odot \Omega^1\mathscr{M})$$

can be regarded as a holomorphic metric on $\mathscr{M}$ which, in fact, reduces to an ordinary holomorphic metric on $\mathscr{M}_{\rm red}$.[5] Since $D(e\otimes\tilde{\varepsilon}) = (De)\otimes\tilde{\varepsilon} + e\otimes(D\tilde{\varepsilon})$, we have further that $Dg = 0$. Together with the condition of vanishing torsion, the proof reduces to that one familiar from ordinary Riemannian geometry (modulo changes of signs due to the $\mathbb{Z}_2$-grading).

Next one realizes that

$$\mathrm{Ber}\,\mathscr{H} \;\cong\; (\mathrm{Ber}\,\Lambda^2\mathscr{H})^{1/(1-\mathcal{N})},$$

which can be deduced from the definition of the Berezinian sheaf by using splitting principle arguments, for instance. Hence,

$$\mathrm{Ber}\,\mathscr{H}^\vee \;\cong\; (\mathrm{Ber}\,\Lambda^2\mathscr{H}^\vee)^{1/(1-\mathcal{N})} \;\cong\; \mathrm{Ber}\,\widetilde{\mathscr{S}}^\vee.$$

Thus, there exists a unique scale (up to multiplicative constants) where D annihilates both, $e \in H^0(\mathscr{M}, \Lambda^2\mathscr{H}^\vee)$ and $\varepsilon \in H^0(\mathscr{M}, \mathrm{Ber}\,\mathscr{H}^\vee)$. Hence, by the uniqueness (for $\mathcal{N} \neq 4$) shown in Prop. 2.2., D coincides with ∇.

□

Hence, $\mathscr{M}$ equipped with that type of connection is a complex quaternionic RC supermanifold. In full analogy with ordinary Riemannian geometry, we shall refer to this connection as the Levi-Civita connection.

§2.9. $\mathcal{N} = 4$ case. As shown in Prop. 2.2., there is no *unique* connection ∇ for $\mathcal{N} = 4$ which is solely determined by the requirements of having totally trace-free torsion and simultaneously annihilating both volume forms on $\mathscr{H}$ and $\widetilde{\mathscr{S}}$. To jump ahead of our story a bit, working with such a connection would result in a dependence of the supertwistor space $\mathscr{P}$ associated with an RC complex quaternionic supermanifold $\mathscr{M}$ on the chosen scale on the latter. Of course, the definition of $\mathscr{P}$ should only depend on the (super)conformal class of $\mathscr{M}$, that is, it should be independent of the particular scale.

Nevertheless, as seen above, the Levi-Civita connection D will always exist no matter what the chosen value of $\mathcal{N}$ is. Moreover, if $\mathcal{N} = 4$, it is possible to compute the change of the Levi-Civita connection under superconformal rescalings since the usual torsion obstructions disappear.

[5]Since e and $\tilde{\varepsilon}$ are assumed to be non-degenerate, their corresponding matrix representations are of full rank and hence as matrices they are invertible.

Proposition 2.5. *Let $\mathscr{M}$ be a $(4|8)$-dimensional RC supermanifold equipped with the Levi-Civita connection. Under a rescaling of the form $\tilde{\varepsilon} \mapsto \gamma\tilde{\varepsilon}$, where γ is a nonvanishing holomorphic function, the change of the Levi-Civita connection D to the new one $\widehat{D}$ is given by the following contorsion tensors:*

$$\Theta_{A\dot{\alpha}B}{}^{C} \;=\; -\gamma_{A\dot{\alpha}}\delta_B{}^C \qquad \text{and} \qquad \Theta_{A\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}} \;=\; \gamma_{A\dot{\alpha}}\delta_{\dot{\beta}}{}^{\dot{\gamma}} \tag{2.40}$$

Here, $\gamma_{A\dot{\alpha}} := E_{A\dot{\alpha}}\log\gamma$, as before. This implies that the new connection $\widehat{D}_{\widehat{A}\widehat{\dot{\alpha}}} = E_{\widehat{A}\widehat{\dot{\alpha}}} \lrcorner \widehat{D} = E_{A\dot{\alpha}} \lrcorner \widehat{D}$ acts as follows:

$$\begin{aligned}
D_{A\dot{\alpha}}\mu^B \;&\mapsto\; \widehat{D}_{\widehat{A}\widehat{\dot{\alpha}}}\mu^{\widehat{B}} \;=\; D_{A\dot{\alpha}}\mu^B - \gamma_{A\dot{\alpha}}\mu^B,\\
D_{A\dot{\alpha}}\lambda^{\dot{\beta}} \;&\mapsto\; \widehat{D}_{\widehat{A}\widehat{\dot{\alpha}}}\lambda^{\widehat{\dot{\beta}}} \;=\; D_{A\dot{\alpha}}\lambda^{\dot{\beta}} + \gamma_{A\dot{\alpha}}\lambda^{\dot{\beta}},\\
D_{A\dot{\alpha}}\mu_B \;&\mapsto\; \widehat{D}_{\widehat{A}\widehat{\dot{\alpha}}}\mu_{\widehat{B}} \;=\; D_{A\dot{\alpha}}\mu_B + \gamma_{A\dot{\alpha}}\mu_B,\\
D_{A\dot{\alpha}}\lambda_{\dot{\beta}} \;&\mapsto\; \widehat{D}_{\widehat{A}\widehat{\dot{\alpha}}}\lambda_{\widehat{\dot{\beta}}} \;=\; D_{A\dot{\alpha}}\lambda_{\dot{\beta}} - \gamma_{A\dot{\alpha}}\lambda_{\dot{\beta}},
\end{aligned} \tag{2.41}$$

where μ^A and $\lambda^{\dot{\alpha}}$ are sections of the vector bundles $\mathscr{H}$ and $\widetilde{\mathscr{S}}$, respectively, together with $\mu^{\widehat{A}} = \mu^A$ and $\lambda^{\widehat{\dot{\alpha}}} = \lambda^{\dot{\alpha}}$ and similarly for their duals.

Proof: Under a change of scale $\tilde{\varepsilon} \mapsto \widehat{\tilde{\varepsilon}} = \gamma\tilde{\varepsilon}$, the symplectic two-form $e \in H^0(\mathscr{M}, \mathrm{Ber}\,\Lambda^2\mathscr{H}^\vee)$ behaves as

$$e \;\mapsto\; \widehat{e} \;=\; \gamma^{\frac{2}{2-\mathcal{N}}}e \;=\; \gamma^{-1}e.$$

This simply follows from the isomorphisms $\mathrm{Ber}\,\mathscr{H}^\vee \cong (\mathrm{Ber}\,\Lambda^2\mathscr{H}^\vee)^{1/(1-\mathcal{N})} \cong \mathrm{Ber}\,\widetilde{\mathscr{S}}^\vee$. Hence, $g = e \otimes \tilde{\varepsilon} \mapsto \widehat{g} = g$. Note that κ (see Eq. (2.21)) vanishes identically for $\mathcal{N} = 4$. In addition, we have

$$\widehat{D}\widehat{e} \;=\; 0 \;=\; \widehat{D}\widehat{\tilde{\varepsilon}}.$$

Hence, the induced contorsion tensor $\Theta_{A\dot{\alpha}B\dot{\beta}}{}^{C\dot{\gamma}}$ must be zero, i.e. $\widehat{D} = D$ upon action on $T\mathscr{M}$ (this result was already expected by Eqs. (2.14) and (2.15) for $\mathcal{N} = 4$). It is then rather straightforward to verify that the above conditions imply Eqs. (2.40).

□

§2.10. Curvature. Given any connection ∇ on $\mathscr{M}$, the associated curvature two-form

$$R \;=\; (R_{\mathbf{A}}{}^{\mathbf{B}}) \;=\; (\tfrac{1}{2}E^{\mathbf{D}} \wedge E^{\mathbf{C}}R_{\mathbf{CDA}}{}^{\mathbf{B}}), \tag{2.42}$$

which takes values in $\mathfrak{g}$, is defined by

$$R_{\mathbf{A}}{}^{\mathbf{B}} \;=\; \mathrm{d}\Omega_{\mathbf{A}}{}^{\mathbf{B}} + \Omega_{\mathbf{A}}{}^{\mathbf{C}} \wedge \Omega_{\mathbf{C}}{}^{\mathbf{B}}. \tag{2.43}$$

The components of the curvature read explicitly as

$$\begin{aligned}
R_{\mathbf{ABC}}{}^{\mathbf{D}} \;&= E_{\mathbf{A}}\Omega_{\mathbf{BC}}{}^{\mathbf{D}} - (-)^{p_{\mathbf{A}}p_{\mathbf{B}}}E_{\mathbf{B}}\Omega_{\mathbf{AC}}{}^{\mathbf{D}} + (-)^{p_{\mathbf{A}}(p_{\mathbf{B}}+p_{\mathbf{C}}+p_{\mathbf{E}})}\Omega_{\mathbf{BC}}{}^{\mathbf{E}}\Omega_{\mathbf{AE}}{}^{\mathbf{D}} - \\
&\quad - (-)^{p_{\mathbf{B}}(p_{\mathbf{C}}+p_{\mathbf{E}})}\Omega_{\mathbf{AC}}{}^{\mathbf{E}}\Omega_{\mathbf{BE}}{}^{\mathbf{D}} - f_{\mathbf{AB}}{}^{\mathbf{E}}\Omega_{\mathbf{EC}}{}^{\mathbf{D}}.
\end{aligned} \tag{2.44}$$

In addition, torsion and curvature are combined into the standard formula

$$[\nabla_{\mathbf{A}}, \nabla_{\mathbf{B}}\} u^{\mathbf{D}} \;=\; (-)^{p_{\mathbf{C}}(p_{\mathbf{A}}+p_{\mathbf{B}})} u^{\mathbf{C}} R_{\mathbf{ABC}}{}^{\mathbf{D}} - T_{\mathbf{AB}}{}^{\mathbf{C}} \nabla_{\mathbf{C}} u^{\mathbf{D}}. \tag{2.45}$$

Here, $u^{\mathbf{A}}$ is some tangent vector on $\mathscr{M}$. This equation might concisely be rewritten as

$$[\nabla_{\mathbf{A}}, \nabla_{\mathbf{B}}\} \;=\; R_{\mathbf{AB}} - T_{\mathbf{AB}}{}^{\mathbf{C}} \nabla_{\mathbf{C}}, \tag{2.46}$$

where $R_{\mathbf{AB}} = R_{\mathbf{AB}C}{}^{D} M_D{}^C + R_{\mathbf{AB}\dot\gamma}{}^{\dot\delta} M_{\dot\delta}{}^{\dot\gamma}$ together with the generators $M_A{}^B$ and $M_{\dot\alpha}{}^{\dot\beta}$ of the Lie superalgebra $\mathfrak{g}$.

Note that because of the factorization $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$, we have

$$R \;=\; R_{\mathscr{H}} \otimes \mathrm{id}_{\widetilde{\mathscr{S}}} + \mathrm{id}_{\mathscr{H}} \otimes R_{\widetilde{\mathscr{S}}}. \tag{2.47}$$

Here, $R_{\mathscr{H}}$ can be viewed as a section of $\Lambda^2\Omega^1\mathscr{M} \otimes \mathrm{End}\,\mathscr{H}$ while $R_{\widetilde{\mathscr{S}}}$ as a section of $\Lambda^2\Omega^1\mathscr{M} \otimes \mathrm{End}\,\widetilde{\mathscr{S}}$. In the structure frame, the decomposition of R looks as

$$R_{\mathbf{A}}{}^{\mathbf{B}} = R_{A\dot\alpha}{}^{B\dot\beta} \;=\; R_A{}^B \delta_{\dot\alpha}{}^{\dot\beta} + \delta_A{}^B R_{\dot\alpha}{}^{\dot\beta}. \tag{2.48}$$

Furthermore, recalling Eq. (2.32), we have further decompositions of R (respectively, of $R_{\mathscr{H}}$ and $R_{\widetilde{\mathscr{S}}}$) into $R^{\mp}$ (respectively, into $R^{\mp}_{\mathscr{H}}$ and $R^{\mp}_{\widetilde{\mathscr{S}}}$).

Proposition 2.6. *Let $\mathscr{M}$ be a complex quaternionic RC supermanifold. In the structure frame, the curvature parts $R^{\mp}_{\mathscr{H}}$ and $R^{\mp}_{\widetilde{\mathscr{S}}}$ of $R^{\mp}$ are of the following form:*

$$\begin{aligned}
R^{-}_{\mathscr{H}} &: R_{A(\dot\alpha B\dot\beta)C}{}^{D} \;=\; -2(-)^{p_C(p_A+p_B)} R_{C[A\dot\alpha\dot\beta}\delta_{B\}}{}^{D},\\
R^{+}_{\mathscr{H}} &: \epsilon_{\dot\alpha\dot\beta} R_{ABC}{}^{D} \;=\; \epsilon_{\dot\alpha\dot\beta}(C_{ABC}{}^{D} - 2(-)^{p_C(p_A+p_B)}\Lambda_{C\{A}\delta_{B]}{}^{D}),\\
R^{-}_{\widetilde{\mathscr{S}}} &: R_{A(\dot\alpha B\dot\beta)\dot\gamma}{}^{\dot\delta} \;=\; C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} + 2\Lambda_{AB}\delta_{(\dot\alpha}{}^{\dot\delta}\epsilon_{\dot\beta)\dot\gamma},\\
R^{+}_{\widetilde{\mathscr{S}}} &: \epsilon_{\dot\alpha\dot\beta} R_{AB\dot\gamma}{}^{\dot\delta},
\end{aligned} \tag{2.49}$$

where $R_{AB\dot\alpha\dot\beta} := R_{AB\dot\alpha}{}^{\dot\gamma}\epsilon_{\dot\gamma\dot\beta}$ and

$$C_{ABC}{}^{D} \;=\; C_{\{ABC]}{}^{D}, \quad (-)^C C_{ABC}{}^{C} \;=\; 0, \quad \Lambda_{AB} \;=\; \Lambda_{[AB\}}, \quad R_{AB\dot\alpha\dot\beta} \;=\; R_{\{AB](\dot\alpha\dot\beta)},$$
$$C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} \;=\; C_{[AB\}(\dot\alpha\dot\beta\dot\gamma)}{}^{\dot\delta}, \quad C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\gamma} \;=\; 0.$$

In addition, the Ricci tensor $\mathrm{Ric}_{A\dot\alpha B\dot\beta} := (-)^{p_C+p_Cp_B} R_{A\dot\alpha C\dot\gamma B\dot\beta}{}^{C\dot\gamma}$ is given by

$$\mathrm{Ric}_{A\dot\alpha B\dot\beta} \;=\; -(2-\mathcal{N}) R_{AB\dot\alpha\dot\beta} + (6-\mathcal{N})\Lambda_{AB}\epsilon_{\dot\alpha\dot\beta}, \tag{2.50}$$

where $\mathrm{Ric}_{A\dot\alpha B\dot\beta} = (-)^{p_Ap_B}\mathrm{Ric}_{B\dot\beta A\dot\alpha}$.

Proof: The proof is based on Bianchi identities and certain index symmetries of the curvature tensor. However, the calculations are rather technical and lengthy, and therefore postponed to App. A.

□

In the following, we shall refer to the quantity Λ_{AB} as the *cosmological constant*.

2.3. Self-dual supergravity equations

§2.11. Self-duality. Let $\mathscr{M}$ be a complex quaternionic RC supermanifold which is equipped with the Levi-Civita connection. It is called *self-dual Einstein* if $C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} = 0$ and simultaneously $R_{AB\dot\alpha}{}^{\dot\beta} = 0$.

Definition 2.4. *A complex quaternionic RC supermanifold is said to be complex quaternionic Kähler if it is equipped with the Levi-Civita connection and is also self-dual Einstein.*

If, in addition, the cosmological constant Λ_{AB} vanishes as well, we call $\mathscr{M}$ *self-dual*. In the latter case, the curvature R is of the form $R = R^+_{\mathscr{H}} \otimes \mathrm{id}_{\widetilde{\mathscr{S}}}$, i.e.

$$[D_{A\dot\alpha}, D_{B\dot\beta}\} \;=\; \epsilon_{\dot\alpha\dot\beta} R_{AB}, \tag{2.51}$$

where R_{AB} is of the form $R_{AB} = C_{ABC}{}^D M_D{}^C$. Furthermore, the connection D has components

$$D_{A\dot\alpha} \;=\; E_{A\dot\alpha}{}^{M\dot\beta}\partial_{M\dot\beta} + \Omega_{A\dot\alpha B}{}^C M_C{}^B. \tag{2.52}$$

Obviously, this says that D on $\widetilde{\mathscr{S}}$ of $T\mathscr{M} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$ is flat. It should be noticed that the superfield components of R_{AB} are not independent of each other because of the Bianchi identities

$$D_{[A\dot\alpha} R_{B\}C} \;=\; 0. \tag{2.53}$$

The field equations of self-dual supergravity with vanishing cosmological constant then follow from these identities together with (2.51). Their explicit form can be found in Siegel [47]. We may summarize by giving the following definition:

Definition 2.5. *A complex quaternionic Kähler RC supermanifold is called a complex hyper-Kähler RC supermanifold if the Levi-Civita connection on $\widetilde{\mathscr{S}}$ is flat.*

In addition, Prop. 2.6. shows that if $\mathscr{M}$ is self-dual, it is also Ricci-flat. Altogether, a complex hyper-Kähler RC supermanifold $\mathscr{M}$ is Ricci-flat and has trivial Berezinian sheaf $\mathrm{Ber}(\mathscr{M})$, i.e. it is a *Calabi-Yau* supermanifold. In this respect, it is worth mentioning that contrary to ordinary complex manifolds, complex supermanifolds with trivial Berezinian sheaf do not automatically admit Ricci-flat metrics (see e.g. Refs. [10]). We shall refer to this latter type of supermanifolds as *formal* Calabi-Yau supermanifolds. Furthermore, for an earlier account of hyper-Kähler supermanifolds of dimension $(4k|2k + 2)$, though in a slightly different setting, see Merkulov [55]. See also Lindström et al. [14].

§2.12. Second Plebanski equation. By analyzing the constraint equations (2.51) in a noncovariant gauge called *light-cone gauge*, Siegel [47] achieved reducing them to a single equation on a superfield Θ, which in fact is the supersymmetrized analog of Plebanski's second equation [61]. In particular, in this gauge the vielbeins turn out to be

$$\begin{aligned} E_{A\dot1}{}^{M\dot\beta} \;&=\; \delta_A{}^M \delta_{\dot1}{}^{\dot\beta} + \tfrac{1}{2}(-)^{p_D}\delta_A{}^N \delta_B{}^O(\partial_{N\dot2}\partial_{O\dot2}\Theta)\omega^{BC}\delta_C{}^M\delta_{\dot2}{}^{\dot\beta},\\ E_{A\dot2}{}^{M\dot\beta} \;&=\; \delta_A{}^M\delta_{\dot2}{}^{\dot\beta}, \end{aligned} \tag{2.54}$$

where $(\partial_{M\dot\alpha}) = (\partial_{\mu\dot\alpha}, \partial_{m\dot\alpha})$ with $\partial_{\mu\dot\alpha} := \partial/\partial x^{\mu\dot\alpha}$ and $\partial_{m\dot\alpha} := \partial/\partial\eta^{m\dot\alpha}$ and $\omega^{AB} := (\epsilon^{\alpha\beta}, \delta^{ij})$. By a slight abuse of notation, we shall write $\partial_{A\dot\alpha} \equiv \delta_A{}^M\partial_{M\dot\alpha}$ in the following. Furthermore, the components of the connection one-form in this gauge are given by

$$\begin{aligned} \Omega_{A\dot 1B}{}^C &= -(-)^{p_D}\tfrac12\partial_{A\dot 2}\partial_{B\dot 2}\partial_{D\dot 2}\Theta\,\omega^{DC}, \\ \Omega_{A\dot 2B}{}^C &= 0. \end{aligned} \tag{2.55}$$

The equation Θ is being subject to is then

$$\epsilon^{\dot\alpha\dot\beta}\partial_{A\dot\alpha}\partial_{B\dot\beta}\Theta + \tfrac12(-)^{p_C}(\partial_{A\dot 2}\partial_{C\dot 2}\Theta)\omega^{CD}(\partial_{D\dot 2}\partial_{B\dot 2}\Theta) = 0. \tag{2.56}$$

In summary, the field equations of self-dual supergravity in light-cone gauge are equivalent to (2.56).

§2.13. Another formulation. Subject of this paragraph is to give another (equivalent) formulation of the self-dual supergravity equations with vanishing cosmological constant. The following proposition generalizes results of Mason and Newman [62] to the supersymmetric situation.

Proposition 2.7. *Let $\mathscr{M}$ be a complex quaternionic RC supermanifold which is equipped with the Levi-Civita connection. Suppose further we are given vector fields $V_{A\dot\alpha}$ on $\mathscr{M}$ which obey*

$$[V_{A(\dot\alpha}, V_{B\dot\beta)}\} = 0. \tag{2.57}$$

Then (2.57) *is an equivalent formulation of the self-dual supergravity equations, i.e. given vector fields $V_{A\dot\alpha}$ on $\mathscr{M}$ satisfying* (2.57)*, it is always possible to find frame fields $E_{\mathbf{A}}$ such that the self-dual supergravity equations with zero cosmological constant are satisfied thus making $\mathscr{M}$ into a complex hyper-Kähler RC supermanifold. Conversely, given a complex hyper-Kähler RC supermanifold $\mathscr{M}$, then there will always exist vector fields $V_{A\dot\alpha}$ on $\mathscr{M}$ which satisfy* (2.57).

Proof: In fact, it is not too difficult to see that (2.57) implies the self-dual supergravity equations with $\Lambda_{AB} = 0$. Indeed, by Frobenius' theorem (see e.g. Manin [51] for the case of supermanifolds) we may choose coordinates such that the $V_{A\dot 2}$s become coordinate derivatives, i.e.

$$V_{A\dot 2} = \partial_{A\dot 2}.$$

In addition, by choosing a gauge such that the $V_{A\dot 1}$s take the form

$$V_{A\dot 1} = \partial_{A\dot 1} + \tfrac12(-)^{p_B}(\partial_{A\dot 2}\partial_{B\dot 2}\Theta)\omega^{BC}\partial_{C\dot 2},$$

where Θ is some to be determined superfield, all equations (2.57) but one are identically satisfied. In particular, only $[V_{A\dot 1}, V_{B\dot 1}\} = 0$ gives a nontrivial condition on Θ. In fact, this equation reduces to (2.56). Therefore, taking the vielbeins and the components of the connection one-form as in (2.54) and (2.55), respectively, we arrive at the desired result.

Conversely, given some complex hyper-Kähler RC supermanifold $\mathscr{M}$, the only nonvanishing components of the connection one-form are $\Omega_{A\dot\alpha B}{}^C$. By virtue of the vanishing of the torsion, Eqs. (2.31) imply

$$f_{\mathbf{AB}}{}^{\mathbf{C}} = \Omega_{\mathbf{AB}}{}^{\mathbf{C}} - (-)^{p_{\mathbf{A}}p_{\mathbf{B}}}\Omega_{\mathbf{BA}}{}^{\mathbf{C}}.$$

Since $\Omega_{\mathbf{AB}}{}^{\mathbf{C}} = \Omega_{A\dot{\alpha}B\dot{\beta}}{}^{C\dot{\gamma}} = \delta_{\dot{\beta}}{}^{\dot{\gamma}}\Omega_{A\dot{\alpha}B}{}^{C}$, we find

$$f_{A(\dot{\alpha}B\dot{\beta})}{}^{C\dot{\gamma}} \;=\; \delta_{(\dot{\alpha}}{}^{\dot{\gamma}}\Omega_{[A\dot{\beta})B\}}{}^{C}.$$

By the discussion given in the last paragraph of Sec. 2.3., we know that there exists a gauge in which $\Omega_{[A\dot{\alpha}B\}}{}^{C}$ vanishes. Therefore, there will always exist vector fields $V_{A\dot{\alpha}}$ obeying (2.57). This concludes the proof.

□

3. Twistor theory

Above we have introduced and discussed complex quaternionic Kähler and hyper-Kähler RC supermanifolds by starting from complex quaternionic RC supermanifolds. In this section, we shall be concerned with their twistorial description. We first construct the supertwistor space, denoted by $\mathscr{P}$, of a complex quaternionic RC supermanifold $\mathscr{M}$. However, as in the purely bosonic situation, we shall see that this will only work if one makes certain additional assumptions about the properties of $\mathscr{M}$. Having presented this construction, we then show which additional structures on $\mathscr{P}$ are needed to render $\mathscr{M}$ into a complex hyper-Kähler RC supermanifold. We further give an alternative formulation and eventually conclude this section by introducing the bundle of local supertwistors.

3.1. Supertwistor space ($\mathcal{N} \neq 4$)

§3.1. Conic structures. In order to proceed in finding an appropriate twistor description, so-called *conic structures* appear to be an adequate tool. Let us therefore recall their definition.

Definition 3.1. (Manin [51]) *Let $\mathscr{M}$ be a complex supermanifold with holomorphic tangent bundle $T\mathscr{M}$. A $(p|q)$-conic structure on $\mathscr{M}$ is a closed subsupermanifold $\mathscr{F}$ in the relative Graßmanian $G_{\mathscr{M}}(p|q;T\mathscr{M})$,*

$$G_{\mathscr{M}}(p|q;T\mathscr{M}) \;:=\; \{\textit{rank } p|q \textit{ local direct summands of } T\mathscr{M}\},$$

such that the projection $\pi : \mathscr{F} \to \mathscr{M}$ is a submersion.

Putting it differently, at any point $x \in \mathscr{M}$ such an $\mathscr{F}$ determines a set of $(p|q)$-dimensional tangent spaces in the fibre of $T\mathscr{M}$ over x corresponding to the points $\pi^{-1}(x) \subset G_{\mathscr{M}}(p|q;T_x\mathscr{M})$.

§3.2. β-plane bundle. Having given this definition, we may now introduce a canonical conic structure on a complex quaternionic RC supermanifold $\mathscr{M}$. Recall again from (2.9) that the tangent bundle $T\mathscr{M}$ of $\mathscr{M}$ is of the form

$$0 \longrightarrow \mathscr{E}\otimes\widetilde{\mathscr{S}} \longrightarrow T\mathscr{M} \longrightarrow \mathscr{S}\otimes\widetilde{\mathscr{S}} \longrightarrow 0,$$

where $\mathscr{S}$ and $\widetilde{\mathscr{S}}$ are both of rank $2|0$ and $\mathscr{E}$ is of rank $0|\mathcal{N}$.

Let now $\mathscr{F}$ be the relative projective line bundle $P_{\mathscr{M}}(\widetilde{\mathscr{S}^\vee})$ on $\mathscr{M}$. Then the above sequence induces a canonical $(2|\mathcal{N})$-conic structure on $\mathscr{M}$, that is, an embedding $\mathscr{F} \hookrightarrow G_{\mathscr{M}}(2|\mathcal{N}; T\mathscr{M})$. In local coordinates, it is given by

$$\begin{aligned} \mathscr{F} \;&\to\; G_{\mathscr{M}}(2|\mathcal{N}; T\mathscr{M}),\\ [\lambda_{\dot\alpha}] \;&\mapsto\; \mathscr{D} \;:=\; \langle \lambda^{\dot\alpha} E_{A\dot\alpha}\rangle. \end{aligned} \tag{3.1}$$

Here, $[\lambda_{\dot\alpha}]$ are homogeneous fibre coordinates of $\pi_1 : \mathscr{F} \to \mathscr{M}$ and the $E_{A\dot\alpha}$s are the frame fields on $\mathscr{M}$. This construction leads naturally to the following definition:

Definition 3.2. *A β-surface Σ in a complex quaternionic RC supermanifold $\mathscr{M}$ is a complex subsupermanifold of dimension $(2|\mathcal{N})$ with the property that at each point $x \in \Sigma$, the tangent space $T_x\Sigma$ is spanned by vectors of the form* (3.1)*, where $\lambda_{\dot\alpha}$ is fixed up to rescalings.*

This in particular means that the components of a tangent vector on Σ are always of the form $\mu^A\lambda^{\dot\alpha}$, where μ^A is arbitrary.

It is worth noting that on $\mathscr{M}_{\rm red}$ this notion of β-surfaces reduces to the standard one (see e.g. Refs. [64, 63]). Next we introduce the notion of *right-flatness*.

Definition 3.3. *A complex quaternionic RC supermanifold $\mathscr{M}$ is said to be right-flat, if the $R_{A(\dot\alpha B\dot\beta\,\dot\gamma\dot\delta)}$-components of the curvature tensor vanish.*

Clearly, for $\mathcal{N} = 0$ this reduces to the standard definition of the vanishing of the anti-self-dual part of the Weyl tensor. Now we are in the position to prove the following proposition:

Proposition 3.1. *Let $\mathscr{M}$ be a complex quaternionic RC supermanifold. For any given point of $\mathscr{F} = P_{\mathscr{M}}(\widetilde{\mathscr{S}^\vee})$ there exists a corresponding β-surface Σ in $\mathscr{M}$ if and only if $\mathscr{M}$ is right-flat.*

Putting it differently, the distribution defined by (3.1) is integrable, i.e. closed under the graded Lie bracket, if and only if $\mathscr{M}$ is right-flat.

Proof: It is not too difficult to show that the graded Lie bracket of two vector fields $E_A := \lambda^{\dot\alpha}E_{A\dot\alpha}$ and $E_B := \lambda^{\dot\beta}E_{B\dot\beta}$ is given by

$$[E_A, E_B\} \;=\; 2\big(\lambda^{\dot\alpha}(\nabla_{[A\dot\alpha}\lambda^{\dot\beta})E_{B\}\dot\beta} - \lambda^{\dot\alpha}\lambda^{\dot\beta}T_{A(\dot\alpha B\dot\beta)}{}^{C\dot\gamma}E_{C\dot\gamma} - \lambda^{\dot\alpha}\Omega_{[A\dot\alpha B\}}{}^{C}E_C\big).$$

However, by virtue of the vanishing of the torsion, the second term on the right-hand side of this equation vanishes identically. Hence, the distribution generated by E_A is integrable if and only if

$$\lambda^{\dot\alpha}\lambda^{\dot\beta}\nabla_{A\dot\alpha}\lambda_{\dot\beta} \;=\; 0. \tag{3.2}$$

Since the integrability condition of this equation is equivalent to the vanishing of the curvature components $R_{A(\dot\alpha B\dot\beta\,\dot\gamma\dot\delta)}$, we arrive at the desired result.

A remark is in order: if $\mathscr{M}$ was not complex quaternionic but only equipped with a connection whose torsion is totally trace-free (cf. our discussion given in §2.7.), then the

above requirement of the integrability of the distribution $\mathscr{D}$ would enforce the vanishing of $T_{A(\dot\alpha B\dot\beta)}{}^{C\dot\gamma}$ as the first term appearing in the equation for $[E_A, E_B\}$ is proportional to the trace. Hence, the whole torsion tensor $T_{A\dot\alpha B\dot\beta}{}^{C\dot\gamma}$ would be zero (see Eq. (2.37)). By virtue of Eq. (3.2), $\mathscr{M}$ would then become a right-flat complex quaternionic RC supermanifold.

$\square$

Note that $\lambda_{\dot\alpha}$ can be normalized such that Eq. (3.2) becomes,

$$\lambda^{\dot\alpha}\nabla_{A\dot\alpha}\lambda_{\dot\beta} \;=\; 0, \tag{3.3}$$

i.e. $\lambda_{\dot\alpha}$ is covariantly constant (i.e. $\lambda_{\dot\alpha}$ is an auto-parallel co-tangent spinor) on $\Sigma \hookrightarrow \mathscr{M}$. In addition, we point out that this equation is scale invariant. This follows from the transformation laws (2.39) of the connection under rescalings. For $\mathcal{N}=4$, this equation is not scale invariant as ∇ is not unique. Therefore, its solutions may depend on the chosen scale on $\mathscr{M}$. We shall address this issue in more detail later on.

Following the terminology of Mason and Woodhouse [63], we shall call $\mathscr{F}$ the *β-plane bundle*. We also refer to $\mathscr{F}$ as the *correspondence space*.

§3.3. Supertwistor space. Note that β-surfaces Σ lift into $\mathscr{F}$ and in addition also foliate $\mathscr{F}$. The lift $\widetilde\Sigma$ of Σ is a section of $\mathscr{F}|_\Sigma \to \Sigma$ satisfying Eqs. (3.2). The tangent vector fields on $\widetilde\Sigma$ are then given by

$$\widetilde E_A \;=\; E_A + \lambda^{\dot\alpha}\lambda_{\dot\gamma}\Omega_{A\dot\alpha\dot\beta}{}^{\dot\gamma}\frac{\partial}{\partial\lambda_{\dot\beta}}. \tag{3.4}$$

Therefore, we canonically obtain an integrable rank-$2|\mathcal{N}$ distribution $\mathscr{D}_{\mathscr{F}} \subset T\mathscr{F}$ on the correspondence space generated by the $\widetilde E_A$s, i.e. $\mathscr{D}_{\mathscr{F}} = \langle \widetilde E_A\rangle$. We shall refer to $\mathscr{D}_{\mathscr{F}}$ as the *twistor distribution*. After quotienting $\mathscr{F}$ by the twistor distribution, we end up with the following double fibration:

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathscr{P} & & \mathscr{M} \end{tikzcd} \tag{3.5}$$

Here, $\mathscr{P}$ is a $(3|\mathcal{N})$-dimensional complex supermanifold which we call the *supertwistor space* of $\mathscr{M}$. Note that this construction is well-defined if we additionally assume that $\mathscr{M}$ is *civilized*, that is, $\mathscr{P}$ is assumed to have the same topology as the supertwistor space associated with any convex region in flat superspace $\mathbb{C}^{4|2\mathcal{N}}$. Otherwise, one my end up with non-Hausdorff spaces; see e.g. Ward and Wells [64] and Mason and Woodhouse [63] for a discussion in the purely bosonic situation. Moreover, without this convexity assumption, the Penrose transform, which relates certain cohomology groups on $\mathscr{P}$ to solutions to certain partial differential equations on $\mathscr{M}$, will not be an isomorphism (see also Sec. 3.2.).

By virtue of this double fibration, we have a geometric correspondence between the two supermanifolds $\mathscr{M}$ and $\mathscr{P}$. In particular, any point $x \in \mathscr{M}$ is associated with the set $\pi_2(\pi_1^{-1}(x))$ in $\mathscr{P}$ consisting of all β-surfaces being *incident* with x. Conversely, any point z in supertwistor space $\mathscr{P}$ corresponds to an β-surface $\pi_1(\pi_2^{-1}(z))$ in $\mathscr{M}$. As $\mathscr{F} \to \mathscr{M}$ is a $\mathbb{P}^1$-bundle over $\mathscr{M}$, the submanifolds $\pi_2(\pi_1^{-1}(x))$ are biholomorphically equivalent to $\mathbb{P}^1$ and are parametrized by $x \in \mathscr{M}$.

We may now state the following basic result:

Theorem 3.1. *There is a one-to-one correspondence between:*

(i) civilized right-flat complex quaternionic RC supermanifolds $\mathscr{M}$ of dimension $(4|2\mathcal{N})$ and

(ii) $(3|\mathcal{N})$-dimensional complex supermanifolds $\mathscr{P}$ each containing a family of rational curves biholomorphically equivalent to $\mathbb{P}^1$ and with normal bundle $\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ inside $\mathscr{P}$ described by

$$0 \longrightarrow \Pi\mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^{\mathcal{N}} \longrightarrow \mathscr{N}_{\mathbb{P}^1|\mathscr{P}} \longrightarrow \mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^2 \longrightarrow 0, \tag{3.6}$$

where $\mathscr{O}_{\mathbb{P}^1}(1)$ is the dual tautological ($c_1 = 1$) bundle on $\mathbb{P}^1$ and Π is the Graßmann parity changing functor.

Proof: Let us first show (i) $\rightarrow$ (ii): In fact, we have already seen that for any complex quaternionic RC supermanifold $\mathscr{M}$ with the above properties, there always exists an associated $(3|\mathcal{N})$-dimensional complex supermanifold $\mathscr{P}$ containing holomorphically embedded projective lines $\pi_2(\pi_1^{-1}(x)) \cong \mathbb{P}^1$ for $x \in \mathscr{M}$. It remains to verify that each of it has a normal bundle $\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ of the above type. To show this, we notice that $\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ is described by the exact sequence

$$0 \longrightarrow \mathscr{D}_{\mathscr{F}} \longrightarrow \pi_1^*T\mathscr{M} \longrightarrow \pi_2^*\mathscr{N}_{\mathbb{P}^1|\mathscr{P}} \longrightarrow 0, \tag{3.7}$$

where $\mathscr{D}_{\mathscr{F}}$ is the twistor distribution. Clearly, the distribution $\mathscr{D}_{\mathscr{F}}$ is described by

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^1}(-1)\otimes\mathbb{C}^2 \longrightarrow \mathscr{D}_{\mathscr{F}} \longrightarrow \Pi\mathscr{O}_{\mathbb{P}^1}(-1)\otimes\mathbb{C}^{\mathcal{N}} \longrightarrow 0$$

when restricted to the fibres $\pi_1^{-1}(x)$ of $\mathscr{F} \rightarrow \mathscr{M}$. Furthermore, $\pi_1^*T\mathscr{M}$ is trivial when restricted to $\pi_1^{-1}(x)$. Therefore, the maps of the above sequence are explicitly given by

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathscr{D}_{\mathscr{F}} & \longrightarrow & \pi_1^*T\mathscr{M} & \longrightarrow & \pi_2^*\mathscr{N}_{\mathbb{P}^1|\mathscr{P}} & \longrightarrow 0, \\ & \mu^A & \mapsto & \mu^A\lambda^{\dot\alpha}, & & & \\ & & & u^{A\dot\alpha} & \mapsto & u^{A\dot\alpha}\lambda_{\dot\alpha}, & \end{array}$$

which completes the proof of the direction (i) $\rightarrow$ (ii).

To show the reverse direction (ii) $\rightarrow$ (i), one simply applies a supersymmetric version of Kodaira's theorem of deformation theory (Waintrob [65]). First, one notices that the obstruction group $H^1(\mathbb{P}^1, \mathscr{N}_{\mathbb{P}^1|\mathscr{P}})$ vanishes which follows from the sequence (3.6) and its induced long exact cohomology sequence:

$$\begin{aligned} 0 \longrightarrow H^0(\mathbb{P}^1, \Pi\mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^{\mathcal{N}}) &\longrightarrow H^0(\mathbb{P}^1, \mathscr{N}_{\mathbb{P}^1|\mathscr{P}}) \longrightarrow \\ \longrightarrow H^0(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^2) &\longrightarrow H^1(\mathbb{P}^1, \Pi\mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^{\mathcal{N}}) \longrightarrow \\ \longrightarrow H^1(\mathbb{P}^1, \mathscr{N}_{\mathbb{P}^1|\mathscr{P}}) &\longrightarrow H^1(\mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^2) \longrightarrow 0. \end{aligned}$$

Then there exists a $\dim_{\mathbb{C}} H^0(\mathbb{P}^1, \mathscr{N}_{\mathbb{P}^1|\mathscr{P}}) = 4|0 + 0|2\mathcal{N} = 4|2\mathcal{N}$ parameter family $\mathscr{M}$ of deformations of $\mathbb{P}^1$ inside $\mathscr{P}$.

If we let $\mathscr{F} := \{(z, \pi_2(\pi_1^{-1}(x))) \,|\, z \in \pi_2(\pi_1^{-1}(x)),\ z \in \mathscr{P},\ x \in \mathscr{M}\} \subset \mathscr{P} \times \mathscr{M}$, then $\mathscr{F}$ is a fibration over $\mathscr{M}$. The typical fibres of $\mathscr{F} \to \mathscr{M}$ are complex projective lines $\mathbb{P}^1$. Hence, we obtain a double fibration

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathscr{P} & & \mathscr{M} \end{tikzcd}$$

where the fibres of $\mathscr{F} \to \mathscr{P}$ are $(2|\mathcal{N})$-dimensional complex subsupermanifolds of $\mathscr{M}$.

Let $T\mathscr{F}/\mathscr{P}$ be the relative tangent sheaf on $\mathscr{F}$ given by

$$0 \longrightarrow T\mathscr{F}/\mathscr{P} \longrightarrow T\mathscr{F} \longrightarrow \pi_2^* T\mathscr{P} \longrightarrow 0.$$

Then (see above) we define a vector bundle $\mathscr{N}$ on $\mathscr{F}$ by

$$0 \longrightarrow T\mathscr{F}/\mathscr{P} \longrightarrow \pi_1^* T\mathscr{M} \longrightarrow \mathscr{N} \longrightarrow 0. \tag{3.8}$$

Clearly, the rank of $\mathscr{N}$ is $2|\mathcal{N}$ and furthermore, the restriction of $\mathscr{N}$ to the fibre $\pi_1^{-1}(x)$ of $\mathscr{F} \to \mathscr{M}$ for $x \in \mathscr{M}$ is isomorphic to the pull-back of the normal bundle of the curve $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{P}$. Hence, $\mathscr{N}$ may be identified with $\pi_2^* \mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ and moreover, the relative tangent sheaf $T\mathscr{F}/\mathscr{P}$ with the twistor distribution $\mathscr{D}_{\mathscr{F}}$.

In addition, the bundle $\pi_1 : \mathscr{F} \to \mathscr{M}$ is of the form $P_{\mathscr{M}}(\widetilde{\mathscr{S}}^\vee)$ for some rank $2|0$ vector bundle $\widetilde{\mathscr{S}}$ (determined below) over $\mathscr{M}$. Then we denote by $\mathscr{O}_{\mathscr{F}}(-1)$ the tautological ($c_1 = -1$) bundle on $\mathscr{F}$. It then follows from the above that the direct images[6] $\pi_{1*}(\Omega^1\mathscr{F}/\mathscr{P} \otimes \mathscr{O}_{\mathscr{F}}(-2))$ and $\pi^1_{1*}(\Omega^1\mathscr{F}/\mathscr{P} \otimes \mathscr{O}_{\mathscr{F}}(-2))$ vanish. Therefore, we find that

$$\pi_{1*}(T\mathscr{F}/\mathscr{P}) \;=\; 0 \;=\; \pi^1_{1*}(T\mathscr{F}/\mathscr{P})$$

upon application of Serre duality.[7]

Applying the direct image functor to the sequence (3.8), we obtain

$$0 \longrightarrow \pi_{1*}(T\mathscr{F}/\mathscr{P}) \longrightarrow \pi_{1*}(\pi_1^* T\mathscr{M}) \longrightarrow \pi_{1*}\mathscr{N} \longrightarrow \pi^1_{1*}(T\mathscr{F}/\mathscr{P}),$$

and hence

$$T\mathscr{M} \;\cong\; \pi_{1*}\mathscr{N} \;\cong\; \pi_{1*}(\pi_2^*\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}).$$

Thus, the sequence (3.6) yields

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{1*}(\pi_2^*(\Pi\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^{\mathcal{N}})) & \longrightarrow & T\mathscr{M} & \longrightarrow & \pi_{1*}(\pi_2^*(\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2)) & \longrightarrow & 0 \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \mathscr{E} \otimes \widetilde{\mathscr{S}} & \longrightarrow & T\mathscr{M} & \longrightarrow & \mathscr{S} \otimes \widetilde{\mathscr{S}} & \longrightarrow & 0 \end{array}$$

[6]Given a mapping $\pi : \mathscr{M} \to \mathscr{N}$ of two complex supermanifolds $\mathscr{M}$ and $\mathscr{N}$, the q-th direct image sheaf $\pi^q_*\mathscr{E}$ of a locally free sheaf $\mathscr{E}$ over $\mathscr{M}$ is defined by the presheaf $\mathscr{N} \supset \mathscr{U}$ open $\mapsto H^q(\pi^{-1}(\mathscr{U}), \mathscr{E})$ with the obvious restriction maps. The zeroth direct image sheaf $\pi^0_*\mathscr{E}$ is usually denoted by $\pi_*\mathscr{E}$.

[7]Recall that Serre duality asserts that for any locally free sheaf $\mathscr{E}$ on a compact complex manifold M of dimension d, we have an isomorphism $H^q(M, \mathscr{E}) \cong H^{d-q}(M, \mathscr{K}_M \otimes \mathscr{E}^\vee)$, where $\mathscr{K}_M$ is the canonical sheaf on M.

since the first direct image sheaf $\pi^1_*(\pi_2^*(\Pi\mathscr{O}_{\mathbb{P}^1}(1)\otimes\mathbb{C}^{\mathcal{N}}))$ vanishes (see above). Above, we have introduced

$$\mathscr{S} \;:=\; \pi_{1*}(\pi_2^*(\mathscr{O}_{\mathbb{P}^1}\otimes\mathbb{C}^2), \quad \widetilde{\mathscr{S}} \;:=\; \pi_{1*}(\pi_2^*(\mathscr{O}_{\mathbb{P}^1}(1)) \quad \text{and} \quad \mathscr{E} \;:=\; \pi_{1*}(\pi_2^*(\Pi\mathscr{O}_{\mathbb{P}^1}\otimes\mathbb{C}^{\mathcal{N}})$$

Notice that by construction, the bundle $\mathscr{F}\cong P_{\mathscr{M}}(\widetilde{\mathscr{S}}^\vee)$ is an integrable $(2|\mathcal{N})$-conic structure on $\mathscr{M}$.

□

§3.4. Gindikin's two-forms and self-dual supergravity with $\Lambda_{AB}=0$. In this and the subsequent paragraph, we shall determine the structure on the supertwistor space $\mathscr{P}$ corresponding to a hyper-Kähler structure on $\mathscr{M}$. In view of that, recall that there always exists a scale where the Levi-Civita connection D coincides with the connection ∇.

Let $E^{A\dot\alpha}$ be the coframe fields on some complex quaternionic Kähler supermanifold $\mathscr{M}$. On the correspondence space $\mathscr{F}$ of the double fibration (3.5), we may introduce a differential two-form $\Sigma(\lambda)$ by setting

$$\Sigma(\lambda) \;:=\; E^{B\dot\beta}\wedge E^{A\dot\alpha}\, e_{AB}\lambda_{\dot\alpha}\lambda_{\dot\beta}, \tag{3.9}$$

where $e\in H^0(\mathscr{M},\Lambda^2\mathscr{H}^\vee)$ is assumed to be non-degenerate and to obey $\nabla e=0$. Furthermore, let d_h be the exterior derivative on $\mathscr{F}$ holding $\lambda_{\dot\alpha}$ constant.

Proposition 3.2. *There is a one-to-one correspondence between gauge equivalence classes of solutions to the self-dual supergravity equations* (2.51) *with vanishing cosmological constant on $\mathscr{M}$ and equivalence classes of (global) d_h-closed non-degenerate differential two-forms $\Sigma(\lambda)$ of the form* (3.9) *on the correspondence space $\mathscr{F}$.*

Proof: First, let us define a differential two-form $\Sigma^{AB}(\lambda)$ by setting

$$\Sigma^{AB}(\lambda) \;:=\; \lambda_{\dot\alpha}\lambda_{\dot\beta}\, E^{A\dot\alpha}\wedge E^{B\dot\beta}.$$

It then follows that $\mathrm{d}_h\Sigma^{AB}$ is given by

$$\mathrm{d}_h\Sigma^{AB} \;=\; -2\lambda_{\dot\alpha}\lambda_{\dot\beta}\, E^{[A\dot\alpha}\wedge \mathrm{d}\, E^{B\}\dot\beta},$$

where d is the exterior derivative on $\mathscr{M}$. Assuming the vanishing of the torsion and upon substituting Eqs. (2.30) into this equation, we see that the connection one-form on $\mathscr{M}$ will be of the form $\Omega_{\mathbf{A}}{}^{\mathbf{B}}=\Omega_{A\dot\alpha}{}^{B\dot\beta}=\delta_{\dot\alpha}{}^{\dot\beta}\Omega_A{}^B$ if and only if

$$\mathrm{d}_h\Sigma^{AB} \;=\; -2\Sigma^{[AC}\wedge\Omega_C{}^{B\}}.$$

Therefore,

$$\mathrm{d}_h\Sigma \;=\; \mathrm{d}_h(\Sigma^{AB}e_{BA}) \;=\; 0,$$

since $\mathrm{d}e_{AB}-2\Omega_{[A}{}^C e_{CB\}}=0$.

□

The differential two-form $\Sigma(\lambda)$ satisfying the properties stated in the immediately preceding proposition is a supersymmetric extension of the Gindikin two-form [66] (see also Ref. [61]). Note that the twistor distribution $\mathscr{D}_{\mathscr{F}}=\langle\widetilde{E}_A\rangle$ annihilates $\Sigma(\lambda)$, i.e. $\Sigma(\lambda)$ descends down to $\mathscr{P}$ (also $\mathrm{d}\Sigma(\lambda)$ is annihilated by $\mathscr{D}_{\mathscr{F}}$).

§3.5. Supertwistor space for complex hyper-Kähler RC supermanifolds. The question which now arises is how the Gindikin two-form can be obtained from certain data given on the supertwistor space $\mathscr{P}$. In the following, we generalize the results known from the purely bosonic situation (see Penrose [21] and also Alekseevsky and Graev [67]).

Let us assume that the supertwistor space $\mathscr{P}$ is a holomorphic fibre bundle $\pi : \mathscr{P} \to \mathbb{P}^1$ over the Riemann sphere $\mathbb{P}^1$. Later on, in §3.9. we shall see that this condition arises quite naturally. Furthermore, let us consider the line bundle $\mathscr{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1 \hookrightarrow \mathscr{P}$ together with its pull-back $\pi^*\mathscr{O}_{\mathbb{P}^1}(2) \to \mathscr{P}$ to $\mathscr{P}$.[8] In addition, let $\Omega^1\mathscr{P}/\mathbb{P}^1$ be the sheaf of relative differential one-forms on $\mathscr{P}$ described by

$$0 \longrightarrow \pi^*\Omega^1\mathbb{P}^1 \longrightarrow \Omega^1\mathscr{P} \longrightarrow \Omega^1\mathscr{P}/\mathbb{P}^1 \longrightarrow 0. \tag{3.10}$$

According to Alekseevsky and Graev [67], we give the following definition (already adopted to our situation):

Definition 3.4. *A section* $\omega \in H^0(\mathscr{P}, \Lambda^2(\Omega^1\mathscr{P}/\mathbb{P}^1)\otimes\pi^*\mathscr{O}_{\mathbb{P}^1}(2))$ *of the sheaf* $\Lambda^2(\Omega^1\mathscr{P}/\mathbb{P}^1)\otimes \pi^*\mathscr{O}_{\mathbb{P}^1}(2)$ *is called a holomorphic relative symplectic structure of type* $\mathscr{O}_{\mathbb{P}^1}(2)$ *on* $\mathscr{P}$ *if it is closed and non-degenerate on the fibres. The integer* $\deg(\mathscr{O}_{\mathbb{P}^1}(2)) = c_1(\mathscr{O}_{\mathbb{P}^1}(2)) = 2$ *is called the weight of* ω*.*[9]

Then Gindikin's two-form $\Sigma(\lambda)$ on $\mathscr{F}$ can be obtained by pulling back the relative symplectic structure ω on $\mathscr{P}$ to the correspondence space $\mathscr{F}$ (and by dividing it by a constant section of $\pi^*\mathscr{O}_{\mathbb{P}^1}(2)$).

Altogether, we may now summarize all the findings from above by stating the following theorem:

Theorem 3.2. *There is a one-to-one correspondence between civilized RC supermanifolds* $\mathscr{M}$ *of dimension* $(4|2\mathcal{N})$ *which are equipped with a hyper-Kähler structure and complex supermanifolds* $\mathscr{P}$ *of dimension* $(3|\mathcal{N})$ *such that:*

(i) $\mathscr{P}$ *is a holomorphic fibre bundle* $\pi : \mathscr{P} \to \mathbb{P}^1$ *over* $\mathbb{P}^1$*,*
(ii) $\mathscr{P}$ *is equipped with a* $(4|2\mathcal{N})$*-parameter family of sections of* π*, each with normal bundle given by* (3.6) *and*
(iii) there exists a holomorphic relative symplectic structure ω *of weight* 2 *on* $\mathscr{P}$*.*

3.2. Equivalent formulation ($\mathcal{N} \neq 4$)

The purpose of this section is to provide an alternative formulation of our above considerations. In this way, we will also be able to describe the case with nonzero cosmological constant. Here, we are generalizing some of the results of Ward [24], of LeBrun [56, 68], of Bailey and Eastwood [60] and of Merkulov [53, 54].

[8] Recall that $\mathscr{O}_{\mathbb{P}^1}(m) := \mathscr{O}_{\mathbb{P}^1}(1)^{\otimes m}$.

[9] Notice that for $\mathcal{N} = 0$ a relative differential two-form is automatically relatively closed as in this case the fibres of π are two-dimensional.

Let $\mathscr{M}$ be a civilized right-flat complex quaternionic RC supermanifold with connection ∇. Let further $\mathscr{P}$ be its associated supertwistor space. There exist several natural vector bundles on $\mathscr{P}$ which encode information about the supermanifold $\mathscr{M}$ and about $\mathscr{P}$ itself, respectively. In the sequel, we shall be using the notation

$$\mathscr{E}[m] \ := \ \mathscr{E} \otimes (\mathrm{Ber}\, \widetilde{\mathscr{S}})^{-m} \ \cong \ \mathscr{E} \otimes (\Lambda^2 \widetilde{\mathscr{S}})^{-m} \tag{3.11}$$

for any locally free sheaf $\mathscr{E}$ on $\mathscr{M}$.

§3.6. Universal line bundle. Let us begin by recalling Eq. (3.3). In fact, this equation implies the existence of a natural holomorphic line bundle $\mathscr{L} \to \mathscr{P}$ over $\mathscr{P}$. Following LeBrun's terminology [56], we shall refer to $\mathscr{L}$ as the *universal line bundle*. It is defined as follows. Let $\mathscr{O}_{\mathscr{F}}(-1)$ be again the tautological bundle on $\mathscr{F}$. Furthermore, denote by $\Omega^1\mathscr{F}/\mathscr{P}$ the sheaf of relative differential one-forms on $\mathscr{F}$ described by the sequence

$$0 \ \longrightarrow \ \pi_2^*\Omega^1\mathscr{P} \ \longrightarrow \ \Omega^1\mathscr{F} \ \longrightarrow \ \Omega^1\mathscr{F}/\mathscr{P} \ \longrightarrow \ 0. \tag{3.12}$$

Then we may define the composition

$$\nabla_{T\mathscr{F}/\mathscr{P}} \ : \ \mathscr{O}_{\mathscr{F}}(-1) \ \stackrel{\pi_1^*\nabla}{\longrightarrow} \ \mathscr{O}_{\mathscr{F}}(-1) \otimes \pi_1^*\Omega^1\mathscr{M} \ \stackrel{\mathrm{id}\otimes\mathrm{res}}{\longrightarrow} \ \mathscr{O}_{\mathscr{F}}(-1) \otimes \Omega^1\mathscr{F}/\mathscr{P}, \tag{3.13}$$

where res denotes the restriction of differential one-forms on $\mathscr{F}$ onto the fibres of the projection $\pi_2 : \mathscr{F} \to \mathscr{M}$. The universal line bundle $\mathscr{L}$ is then defined by the zeroth direct image

$$\mathscr{L} \ := \ \pi_{2*}(\ker \nabla_{T\mathscr{F}/\mathscr{P}}). \tag{3.14}$$

Hence, the fibre of $\mathscr{L}$ over a point $z \in \mathscr{P}$ is the space of solutions to $\lambda^{\dot\alpha}\nabla_{A\dot\alpha}\lambda_{\dot\beta} = 0$ on the β-surface $\pi_1(\pi_2^{-1}(z))$. Note that $\mathscr{L}$ restricted to $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{P}$, for $x \in \mathscr{M}$, can be identified with $\mathscr{O}_{\mathbb{P}^1}(-1)$.

§3.7. Jacobi bundle. The second bundle over the supertwistor space we are interested in is the so-called *Jacobi bundle* (see also Refs. [56, 54]). Let us denote it by $\mathscr{J}$. It is defined to be the solution space of the *supertwistor equation*

$$\lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\omega^B + \delta_A{}^B\pi_{\dot\alpha}) \ = \ 0, \tag{3.15}$$

on the β-surface $\pi_1(\pi_2^{-1}(z))$. Here, $\lambda_{\dot\alpha}$ is non-zero and obeys (3.3) and $\pi_{\dot\alpha}$ is arbitrary. Note that (3.15) does not depend on the chosen scale on $\mathscr{M}$. Note further that the rank of $\mathscr{J}$ is $3|\mathcal{N}$. Then we have the following result:

Proposition 3.3. *There is a natural isomorphism $T\mathscr{P} \cong \mathscr{J} \otimes \mathscr{L}^{-1}$.*

Proof: Let z be a point in $\mathscr{P}$ and $\Sigma := \pi_1(\pi_2^{-1}(z))$ the associated β-surface in $\mathscr{M}$, and let $\lambda^{\dot\alpha}$ be a section of $\mathscr{L}$.

It is always possible to have a one-parameter foliation of Σ since

$$\lambda_{\dot\beta}\lambda^{\dot\alpha}\nabla_{A\dot\alpha}(\mu^B\lambda^{\dot\beta}) \ = \ \lambda_{\dot\beta}\lambda^{\dot\beta}\lambda^{\dot\alpha}\nabla_{A\dot\alpha}\mu^B \ = \ 0,$$

where $\mu^A\lambda^{\dot\alpha}$ is any tangent vector field to Σ. Let now $J = J^{A\dot\alpha}E_{A\dot\alpha}$ be the associated Jacobi field on Σ. Then any tangent vector $(\omega^A, \pi_{\dot\alpha})$ at $z \in \mathscr{P}$ can be represented by Jacobi fields on Σ,

$$\omega^A \ = \ J^{A\dot\alpha}\lambda_{\dot\alpha} \qquad \text{and} \qquad \pi_{\dot\alpha} \ = \ J^{B\dot\beta}\nabla_{B\dot\beta}\lambda_{\dot\alpha},$$

subject to the constraint

$$\mathcal{L}_J X \bmod T\Sigma \ = \ [J, X\} \bmod T\Sigma \ = \ 0 \qquad \text{for any} \qquad X \in T\Sigma.$$

Note that the above equations are unaffected by changes of the form $J^{A\dot\alpha} \mapsto J^{A\dot\alpha} + J^A\lambda^{\dot\alpha}$, where $J^A\lambda^{\dot\alpha}$ is also a Jacobi field which is in addition tangent to Σ. Therefore, tangent vectors at $z \in \mathscr{P}$ are actually represented by *equivalence classes* of Jacobi fields, where two Jacobi fields are said to be equivalent if their difference lies in $T\Sigma$.

Explicitly, the constraint $[J, X\} \in T\Sigma$ reads as

$$\lambda_{\dot\beta}(\lambda^{\dot\alpha}\nabla_{A\dot\alpha}J^{B\dot\beta} - \delta_A{}^B J^{C\dot\gamma}\nabla_{C\dot\gamma}\lambda^{\dot\beta}) \ = \ 0. \tag{3.16}$$

Using this expression, one may straightforwardly check that ω^A obeys the supertwistor equation (3.15). Hence, the mapping $J \otimes \lambda \mapsto \omega$ defines a morphism $T\mathscr{P} \otimes \mathscr{L} \to \mathscr{J}$. Since the solution space of (3.15) is of the right dimensionality, we have thus constructed an isomorphism.

□

§3.8. Einstein bundle. The last vector bundle we are about to define is the *Einstein bundle*. Originally, it was introduced by LeBrun [56] in the context of the ambitwistor space (the space of complex null-geodesics of some given complex four-dimensional manifold) and its relation to the (full) Einstein equations. He showed that non-vanishing sections of this bundle are in one-to-one correspondence with Einstein metrics in the given conformal class. Unfortunately, the Einstein bundle on ambitwistor space and its generalization to superambitwistor space in the context of $\mathcal{N} = 1$ supergravity (cf. Merkulov [54]) seem only to be definable in terms of their inverse images on the associated correspondence space, that is, so far it lacks a description in terms of the intrinsic structure of the (super)ambitwistor space. As we shall see in a moment, this will not be the case if the (super)manifold under consideration is (super)conformally right-flat. It is this additional condition that allows for giving an explicit description of this bundle in terms of natural holomorphic sheaves on the (super)twistor space. As we shall see, this bundle will also yield a reinterpretation of the results given in Thm. 3.2. Our subsequent discussion is a generalization of the ideas of [56, 60, 53, 54].

Next we introduce a second-order differential operator, Δ, on the correspondence space $\mathscr{F}$ which is given in the structure frame by

$$\Delta_{AB} \ := \ \lambda^{\dot\alpha}\lambda^{\dot\beta}(\nabla_{\{A\dot\alpha}\nabla_{B]\dot\beta} + R_{AB\dot\alpha\dot\beta}), \tag{3.17}$$

where $\lambda_{\dot\alpha}$ obeys (3.3) and $R_{AB\dot\alpha\dot\beta}$ is the $R^+_{\mathscr{S}}$-part of the curvature as discussed in Prop. 2.6. It then follows that Δ is independent of the choice of scale if it acts on sections of $\pi_1^{-1}\mathscr{O}_{\mathscr{M}}[-1]$.

This can be seen as follows: let φ be a section of $\mathscr{O}_{\mathscr{M}}[k]$. If one performs a change of scale according to $\tilde{\varepsilon} \mapsto \gamma\tilde{\varepsilon}$, the connection changes as follows:

$$\widehat{\nabla}_{\widehat{A}\widehat{\dot{\alpha}}}\varphi \;=\; \gamma^{-\kappa}(\nabla_{A\dot{\alpha}}\varphi - k\gamma_{A\dot{\alpha}}\varphi), \tag{3.18}$$

where κ was defined in (2.21) and $\gamma_{A\dot{\alpha}} := E_{A\dot{\alpha}}\log\gamma$, as before. Therefore, if one chooses $k = -1$ one arrives after a few lines of algebra at

$$\widehat{\nabla}_{\{\widehat{A}(\widehat{\dot{\alpha}}}\widehat{\nabla}_{\widehat{B}]\widehat{\dot{\beta}})}\varphi \;=\; \gamma^{-2\kappa}(\nabla_{\{A(\dot{\alpha}}\nabla_{B]\dot{\beta})}\varphi + \nabla_{\{A(\dot{\alpha}}\gamma_{B]\dot{\beta})}\varphi - \gamma_{\{A(\dot{\alpha}}\gamma_{B]\dot{\beta})}\varphi). \tag{3.19}$$

In a similar manner, one may verify that

$$\widehat{R}_{\widehat{A}\widehat{B}\widehat{\dot{\alpha}}\widehat{\dot{\beta}}} \;=\; \gamma^{-2\kappa}(R_{AB\dot{\alpha}\dot{\beta}} - \nabla_{\{A(\dot{\alpha}}\gamma_{B]\dot{\beta})} + \gamma_{\{A(\dot{\alpha}}\gamma_{B]\dot{\beta})}). \tag{3.20}$$

Combining these two expressions, one arrives at the desired result.

As Δ acts on the fibres of $\pi_2 : \mathscr{F} \to \mathscr{P}$, we can define the Einstein bundle $\mathscr{E}$ on $\mathscr{P}$ by the following resolution:

$$0 \;\longrightarrow\; \pi_2^{-1}\mathscr{E} \;\longrightarrow\; \pi_1^{-1}\mathscr{O}_{\mathscr{M}}[-1] \;\stackrel{\Delta}{\longrightarrow}\; \pi_1^*(\odot^2\mathscr{H}^\vee[-3]) \otimes \mathscr{O}_{\mathscr{F}}(2) \;\longrightarrow\; 0, \tag{3.21}$$

where $\mathscr{O}_{\mathscr{F}}(2)$ is the second tensor power of the dual of the tautological bundle $\mathscr{O}_{\mathscr{F}}(-1)$ on the correspondence space $\mathscr{F}$.

We are now in the position to relate the four bundles $T\mathscr{P}$, $\mathscr{L}$, $\mathscr{J}$ and $\mathscr{E}$ among themselves by virtue of the following proposition:

Proposition 3.4. *There is a natural isomorphism of sheaves* $\mathscr{E} \cong \Omega^1\mathscr{P}\otimes\mathscr{L}^{-2} \cong \mathscr{J}^\vee\otimes\mathscr{L}^{-1}$.

Proof: The second isomorphism is the one proven in Prop. 3.3. So it remains to verify the first one. Recall again that $T\mathscr{P} \cong \mathscr{J} \otimes \mathscr{L}^{-1}$, that is, the fibre of $T\mathscr{P}$ over some point $z \in \mathscr{P}$ is the space of solutions of the supertwistor equation on $\pi_2^{-1}(z)$ for ω^A being of homogeneous degree one in $\lambda_{\dot{\alpha}}$. The fibre of the Einstein bundle $\mathscr{E}$ over $z \in \mathscr{P}$ coincides with the kernel of Δ on the same subsupermanifold $\pi_2^{-1}(z) \hookrightarrow \mathscr{F}$.

Consider now the scalar

$$Q \;:=\; (2-\mathcal{N})\omega^A\lambda^{\dot{\alpha}}\nabla_{A\dot{\alpha}}\varphi - \varphi(-)^{p_A}\lambda^{\dot{\alpha}}\nabla_{A\dot{\alpha}}\omega^A,$$

where φ is a section of $\pi_1^{-1}\mathscr{O}_{\mathscr{M}}[-1]$ and ω^A a solution to the supertwistor equation (3.15). Clearly, Q is of homogeneous degree two in $\lambda_{\dot{\alpha}}$ and as one may check, it is independent of the choice of scale. In showing the latter statement, one needs the relation

$$\lambda^{\dot{\alpha}}\nabla_{A\dot{\alpha}}\omega^B \;=\; \tfrac{1}{2-\mathcal{N}}\delta_A{}^B(-)^{p_C}\lambda^{\dot{\gamma}}\nabla_{C\dot{\gamma}}\omega^C,$$

which follows from the supertwistor equation (3.15). In addition, upon using the very same equation (3.15) together with $\Delta_{AB}\varphi = 0$, one finds that

$$\lambda^{\dot{\alpha}}\nabla_{A\dot{\alpha}}Q \;=\; 0.$$

Hence, the quantity Q corresponds to a point in the fibre of $\mathscr{L}^{-2}$ over the point $z \in \mathscr{P}$. Altogether, Q provides a non-degenerate $\mathscr{L}^{-2}$-valued pairing of the fibres of tangent bundle $T\mathscr{P}$ and of the Einstein bundle $\mathscr{E}$, thus establishing the claimed isomorphism.

$\square$

This shows, as indicated earlier, that the Einstein bundle is fully determined in terms of the intrinsic structure of the supertwistor space.

§3.9. Hyper-Kähler structures. The next step is to verify the following statement:

Proposition 3.5. *There is a natural one-to-one correspondence between scales on a civilized right-flat complex quaternionic RC supermanifold $\mathscr{M}$ in which the $R^{+}_{\widetilde{\mathscr{S}}}$-part of the curvature vanishes and nonvanishing sections of the Einstein bundle $\mathscr{E}$ over the associated supertwistor space $\mathscr{P}$.*

Putting it differently, nonvanishing sections of the Einstein bundle are in one-to-one correspondence with (equivalence classes of) solutions to the self-dual supergravity equations with nonzero cosmological constant.

Proof: By our convexity assumption (recall that $\mathscr{M}$ is assumed to be civilized; putting it differently, there is a Stein covering of $\mathscr{M}$), we have $H^q(\pi_2^{-1}(z), \mathbb{C}) \cong 0$ for $z \in \mathscr{P}$ and $q \geq 1$. Let $\mathscr{U} \subset \mathscr{M}$ be an open subset and set $\mathscr{U}' := \pi_1^{-1}(\mathscr{U}) \subset \mathscr{F}$ and $\mathscr{U}'' := \pi_2(\mathscr{U}') \subset \mathscr{P}$. Therefore, we have an isomorphism[10]

$$H^r(\mathscr{U}'', \mathscr{E}) \;\cong\; H^r(\mathscr{U}', \pi_2^{-1}\mathscr{E}).$$

Hence, in order to compute $H^r(\mathscr{U}'', \mathscr{E})$ we need to compute $H^r(\mathscr{U}', \pi_2^{-1}\mathscr{E})$. However, the latter cohomology groups can be computed from the exact resolution (3.21) upon applying the direct image functor

$$0 \longrightarrow \pi_{1*}(\pi_2^{-1}\mathscr{E}) \longrightarrow \pi_{1*}\mathscr{R}^0 \longrightarrow \pi_{1*}\mathscr{R}^1 \longrightarrow \pi^1_{1*}(\pi_2^{-1}\mathscr{E})$$

where we have abbreviated

$$\mathscr{R}^0 \;:=\; \pi_1^{-1}\mathscr{O}_{\mathscr{M}}[-1] \qquad \text{and} \qquad \mathscr{R}^1 \;:=\; \pi_1^*(\odot^2\mathscr{H}^\vee[-3]) \otimes \mathscr{O}_{\mathscr{F}}(2).$$

In addition, there is a spectral sequence converging to

$$H^{p+q}(\mathscr{U}', \pi_2^{-1}\mathscr{E}),$$

with

$$E_1^{p,q} \;\cong\; H^0(\mathscr{U}, \pi^q_{1*}\mathscr{R}^p).$$

Notice the sheaves in resolution have vanishing higher direct images while the zeroth images are given by

$$\begin{aligned} \pi_{1*}\mathscr{R}^0 \;&\cong\; \mathscr{O}_{\mathscr{M}}[-1], \\ \pi_{1*}\mathscr{R}^1 \;&\cong\; (\odot^2\mathscr{H}^\vee \otimes \odot^2\widetilde{\mathscr{S}}^\vee)[-1]. \end{aligned}$$

[10]Note that this in fact holds true for any locally free sheaf on the supertwistor space.

Therefore, the cohomology group $H^0(\mathscr{U}'', \mathscr{E}) \cong H^0(\mathscr{U}', \pi_2^{-1}\mathscr{E})$ is isomorphic to the kernel of a second order differential operator which by virtue of our above discussion turns out to be the solution space of

$$(\nabla_{\{A(\dot\alpha}\nabla_{B]\dot\beta)} + R_{AB\dot\alpha\dot\beta})\varphi \;=\; 0,$$

where φ is a nonvanishing section of $\mathscr{O}_{\mathscr{M}}[-1]$. As discussed above, this equation is independent of the choice of scale on $\mathscr{M}$. Since φ is a nonvanishing section of $\mathscr{O}_{\mathscr{M}}[-1]$, we may always work in the scale where $\varphi = 1$. Thus, the above equation implies that $R_{AB\dot\alpha\dot\beta}$ must vanish and the proof is completed.

□

Let now τ be the section of $\mathscr{E}$ corresponding to the scale where $R^+_{\mathscr{S}}$ vanishes. Obviously, it defines a $(2|\mathcal{N})$-dimensional distribution on the supertwistor space $\mathscr{P}$ given by the kernel of τ. One also says that τ is a non-degenerate holomorphic contact form determing a holomorphic contact structure (a distribution of Graßmann even codimension one) on $\mathscr{P}$. Thus, non-degeneracy of τ insures a nonvanishing cosmological constant. In addition, let ∇ be the connection on $\mathscr{M}$ defined by this chosen scale. In the remainder, we shall show that degenerate contact structures on $\mathscr{P}$ are in one-to-one correspondence with (equivalence classes of) solutions to the self-dual supergravity equations with zero cosmological constant.

Proposition 3.6. *The $(4|2\mathcal{N})$-dimensional distribution on $\mathscr{F}$ defined by $\pi_2^*\tau$ coincides with the $(4|2\mathcal{N})$-dimensional distribution defined by $\pi_1^*\nabla$.*

Proof: Recall that $\mathscr{E} \cong \Omega^1\mathscr{P}\otimes\mathscr{L}^{-2}$. Then we note that the pairing $\Omega^1\mathscr{P}\otimes\mathscr{L}^{-2}\times T\mathscr{P} \to \mathscr{L}^{-2}$ is given by

$$(2-\mathcal{N})\omega^A\lambda^{\dot\alpha}\nabla_{A\dot\alpha}\varphi - \varphi(-)^{p_A}\lambda^{\dot\alpha}\nabla_{A\dot\alpha}\omega^A,$$

as follows by the discussion given in the proof of Prop. 3.4. Here, φ represents τ on $\mathscr{F}$ and ω^A corresponds to a tangent vector on $\mathscr{P}$. In the scale defined by τ, we have $\varphi = 1$ (cf. the proof of the immediately preceding proposition). By virtue of the the supertwistor equation (3.15), we conclude that the distribution on the correspondence space $\mathscr{F}$ defined by the vanishing of this pairing is given by

$$\lambda^{\dot\alpha}\nabla_{A\dot\alpha}\omega^B \;=\; 0.$$

Eq. (3.16) in turn implies that a solution to this equation must correspond to a Jacobi field $J^{A\dot\alpha}$ which satisfies

$$J^{A\dot\alpha}\lambda^{\dot\beta}\nabla_{A\dot\alpha}\lambda_{\dot\beta} \;=\; 0.$$

Therefore, we have a correspondence between subspaces of the fibre of $T\mathscr{P}$ over a point $z \in \mathscr{P}$ which are annihilated by the differential one-form τ and Jacobi fields on the β-surface $\Sigma = \pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathscr{M}$ which are annihilated by the differential one-form $E^{A\dot\alpha}\lambda^{\dot\beta}\nabla_{A\dot\alpha}\lambda_{\dot\beta}$. In fact, this form is the push-forward to $\mathscr{M}$ of the differential one-form on $\mathscr{F}$ defining the distribution given by $\pi_1^*\nabla$.

□

Then we have the following result:

Proposition 3.7. *Let τ be the section of the Einstein bundle $\mathscr{E} \to \mathscr{P}$ corresponding to the scale in which $R^{+}_{\widetilde{\mathscr{S}}}$ vanishes. In this scale, the cosmological constant will vanish if and only if the distribution on $\mathscr{P}$ defined by τ is integrable. Hence, the Ricci tensor is zero.*

This means that nonvanishing integrable sections of the Einstein bundle (that is, degenerate contact structures) are in one-to-one correspondence with (equivalence classes of) solutions to the self-dual supergravity equations with zero cosmological constant.
Proof: Obviously, showing integrability of the distribution on $\mathscr{P}$ defined by τ is equivalent to showing the integrability of the distribution on $\mathscr{F}$ defined by the pull-pack $\pi_2^*\tau$. Prop. 3.6. implies that the distribution defined by $\pi_2^*\tau$ will be integrable if and only if $\widetilde{\mathscr{S}^\vee}$ is projectively flat in the scale defined by τ (see also comment after proof of Prop. 3.1. leading to Eq. (3.3)). We conclude from Prop. 2.6. that in addition to $R_{AB\dot\alpha\dot\beta}$ also Λ_{AB} must vanish. Hence, the Ricci tensor is zero.

□

Recall that τ defines a $(2|\mathcal{N})$-dimensional distribution on $\mathscr{P}$. If this distribution is integrable, it gives a foliation of $\mathscr{P}$ by $(2|\mathcal{N})$-dimensional subsupermanifolds. In fact, it yields a holomorphic fibration

$$\mathscr{P} \ \to\ \mathbb{P}^1 \tag{3.22}$$

of the supertwistor space over the Riemann sphere (see also Penrose [21] for the purely bosonic situation). Remember that this fibration was one of the assumptions made in Thm. 3.2. Therefore, we may conclude that if the distribution τ is integrable the supertwistor space $\mathscr{P}$ is equipped with a relative symplectic structure as stated in point (iii) of Thm. 3.2.

§3.10. Summary. Let us summarize all the correspondences derived above in the following table:

supertwistor spaces $\mathscr{P}$	civilized right-flat complex quaternionic RC supermanifolds, i.e. $C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} = 0$
supertwistor spaces $\mathscr{P}$ with non-degenerate holomorphic contact structures	civilized right-flat complex quaternionic RC supermanifolds which are self-dual Einstein, i.e. $C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} = 0$ and $R_{AB\dot\alpha}{}^{\dot\beta} = 0$
supertwistor spaces $\mathscr{P}$ with degenerate holomorphic contact structures	civilized right-flat complex quaternionic RC supermanifolds which are self-dual, i.e. $C_{AB\dot\alpha\dot\beta\dot\gamma}{}^{\dot\delta} = 0$, $R_{AB\dot\alpha}{}^{\dot\beta} = 0$ and $\Lambda_{AB} = 0$

We remind the reader that the curvature components can be found in Prop. 2.6.

3.3. Bundle of local supertwistors ($\mathcal{N} \neq 4$)

This subsection is devoted to the bundle of local supertwistors and its implications on the supermanifolds under consideration. Here, we give a generalization of methods developed by Penrose [69], by LeBrun [56] and by Bailey and Eastwood [60]. So, let $\mathscr{M}$ be a civilized right-flat complex quaternionic RC supermanifold with connection ∇, in the sequel.

§3.11. Bundle of local supertwistors. Let us start by recalling the jet sequence (for a proof, see e.g. Manin [51])

$$0 \longrightarrow \Omega^1\mathscr{M}\otimes\mathscr{E} \longrightarrow \mathrm{Jet}^1\mathscr{E} \longrightarrow \mathscr{E} \longrightarrow 0, \tag{3.23}$$

where $\mathscr{E}$ is some locally free sheaf on $\mathscr{M}$ and $\mathrm{Jet}^1\mathscr{E}$ is the sheaf of first-order jets of $\mathscr{E}$. Recall further the factorization of the tangent bundle of $\mathscr{M}$ as $T\mathscr{M} \cong \mathscr{H}\otimes\widetilde{\mathscr{S}}$. Choose now $\mathscr{E}$ to be $\mathscr{H}$. Hence, the above sequence becomes

$$0 \longrightarrow (\mathscr{H}\otimes\mathscr{H}^\vee)\otimes\widetilde{\mathscr{S}}^\vee \longrightarrow \mathrm{Jet}^1\mathscr{H} \longrightarrow \mathscr{H} \longrightarrow 0. \tag{3.24}$$

Since, $(\mathscr{H}\otimes\mathscr{H}^\vee)_0\otimes\widetilde{\mathscr{S}}^\vee$, where $(\mathscr{H}\otimes\mathscr{H}^\vee)_0$ means the trace-free part of $\mathscr{H}\otimes\mathscr{H}^\vee$, is a subbundle of $\Omega^1\mathscr{M}\otimes\mathscr{H}$, i.e.

$$0 \longrightarrow (\mathscr{H}\otimes\mathscr{H}^\vee)_0\otimes\widetilde{\mathscr{S}}^\vee \longrightarrow \Omega^1\mathscr{M}\otimes\mathscr{H}, \tag{3.25}$$

we may define a rank-$4|\mathcal{N}$ bundle, denoted by $\mathscr{T}$, over $\mathscr{M}$ by the following sequence:

$$0 \longrightarrow (\mathscr{H}\otimes\mathscr{H}^\vee)_0\otimes\widetilde{\mathscr{S}}^\vee \longrightarrow \mathrm{Jet}^1\mathscr{H} \longrightarrow \mathscr{T} \longrightarrow 0. \tag{3.26}$$

We shall call $\mathscr{T}$ the *bundle of local supertwistors*. The reason for naming it like this will become clear in due course of our subsequent discussion.

As a first result, we obtain from (3.24) and (3.26) a natural isomorphism:

$$\mathrm{Ber}\,\mathscr{T} \;\cong\; \mathrm{Ber}\,\mathscr{H}\otimes(\mathrm{Ber}\,\widetilde{\mathscr{S}})^{-1}. \tag{3.27}$$

Hence, by virtue of (2.15) we may conclude that

$$\mathrm{Ber}\,\mathscr{T} \;\cong\; \mathscr{O}_{\mathscr{M}}. \tag{3.28}$$

Furthermore, in a structure frame, $\mathscr{T}$ may be described by natural fibre coordinates of the form $(\omega^A, \pi_{\dot\alpha})$. Under a change of scale $\tilde\varepsilon \mapsto \gamma\tilde\varepsilon$, these coordinates behave as

$$\omega^A \;\mapsto\; \widehat{\omega}^{\widehat{A}} \;=\; \gamma^{\frac{1}{2}\kappa}\omega^A \qquad \text{and} \qquad \pi_{\dot\alpha} \;\mapsto\; \widehat{\pi}_{\widehat{\dot\alpha}} \;=\; \gamma^{-\frac{1}{2}\kappa}(\pi_{\dot\alpha} - \omega^A\gamma_{A\dot\alpha}), \tag{3.29}$$

which is an immediate consequence of the transformation laws (2.39). Remember that the constant κ appearing above was introduced in (2.21) and $\gamma_{A\dot\alpha}$ was defined to be $\gamma_{A\dot\alpha} = E_{A\dot\alpha}\log\gamma$. Altogether, these considerations imply that there is a canonical exact sequence

$$0 \longrightarrow \widetilde{\mathscr{S}}^\vee \longrightarrow \mathscr{T} \longrightarrow \mathscr{H} \longrightarrow 0. \tag{3.30}$$

§3.12. Local supertwistor connection. In the class of affine connections on the bundle $\mathscr{T}$, there exists a distinguished one referred to as the *local supertwistor connection*, in the following. This is an immediate consequence of the scaling behavior (3.29), as we shall see now. Let us mention in passing that this particular connection will be unique and independent of the choice of scale on $\mathscr{M}$.

Let us recall the supertwistor equation (3.15), which we repeat for the reader's convenience at this stage

$$\lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\omega^B + \delta_A{}^B\pi_{\dot\alpha}) \;=\; 0. \tag{3.31}$$

Recall further from the proof of Prop. 3.3. that tangent vectors at $z \in \mathscr{P}$ can be represented by $\omega^A = J^{A\dot\alpha}\lambda_{\dot\alpha}$ and $\pi_{\dot\alpha} = J^{B\dot\beta}\nabla_{B\dot\beta}\lambda_{\dot\alpha}$, where $J = J^{A\dot\alpha}E_{A\dot\alpha}$ is a Jacobi field on the β-surface $\pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathscr{M}$. In the very same proof, we have argued that this ω^A satisfies the supertwistor equation. Similarly, one may show that

$$\lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\pi_{\dot\beta} + (-)^{p_B}(R_{AB\dot\alpha\dot\beta} - \Lambda_{AB}\epsilon_{\dot\alpha\dot\beta})\omega^B) \;=\; 0. \tag{3.32}$$

Here, we have made use of the curvature decompositions (2.49). Furthermore, the scaling behavior (3.29) is exactly of the same form as the one of $\omega^A = J^{A\dot\alpha}\lambda_{\dot\alpha}$ and $\pi_{\dot\alpha} = J^{B\dot\beta}\nabla_{B\dot\beta}\lambda_{\dot\alpha}$, respectively. That it is why we have denoted the fibre coordinates of the bundle $\mathscr{T}$ by the same letters.

Altogether, Eqs. (3.31) and (3.32) can be reinterpreted as an $SL(4|\mathcal{N})$-connection $\mathcal{D}$ on $\mathscr{T}$ – the local supertwistor connection:

$$\lambda^{\dot\alpha}\mathcal{D}_{A\dot\alpha}\begin{pmatrix}\omega^B\\ \pi_{\dot\beta}\end{pmatrix} \;=\; \begin{pmatrix}\lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\omega^B + \delta_A{}^B\pi_{\dot\alpha})\\ \lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\pi_{\dot\beta} + (-)^{p_B}(R_{AB\dot\alpha\dot\beta} - \Lambda_{AB}\epsilon_{\dot\alpha\dot\beta})\omega^B)\end{pmatrix}. \tag{3.33}$$

Thus, flat sections with respect to this connection correspond to solutions of the supertwistor equation. After all, this justifies the name local supertwistor bundle.

Let us make the following abbreviations: $\mathcal{D}_A := \lambda^{\dot\alpha}\mathcal{D}_{A\dot\alpha}$, $\nabla_A := \lambda^{\dot\alpha}\nabla_{A\dot\alpha}$ and ${}^tZ := (\omega^A, \pi_{\dot\alpha})$. Then we may rewrite Eqs. (3.33) concisely as

$$\mathcal{D}_A Z \;=\; \nabla_A Z + \mathcal{A}_A Z, \tag{3.34}$$

where

$$\mathcal{A}_A \;:=\; \begin{pmatrix} 0 & \delta_A{}^B\lambda^{\dot\beta}\\ (-)^{p_B}\lambda^{\dot\alpha}(R_{AB\dot\alpha\dot\beta} - \Lambda_{AB}\epsilon_{\dot\alpha\dot\beta}) & 0\end{pmatrix}, \tag{3.35}$$

is the $\mathfrak{sl}(4|\mathcal{N})$-valued gauge potential. The local supertwistor connection $\mathcal{D}$ is torsion-free, since ∇ is torsion-free. Furthermore, the $\mathcal{F}^-$-part of the curvature two-form $\mathcal{F} = \mathcal{D}^2 = \mathcal{F}^- + \mathcal{F}^+$ of $\mathcal{D}$ (here, we are using the notation of Prop. 2.6.) is given in a structure frame by

$$\begin{aligned}\mathcal{F}_{AB} \;&=\; \lambda^{\dot\alpha}\lambda^{\dot\beta}\mathcal{F}_{A\dot\alpha B\dot\beta}\\ &=\; [\mathcal{D}_A, \mathcal{D}_B\}\\ &=\; R_{AB} + \nabla_A\mathcal{A}_B - (-)^{p_A p_B}\nabla_B\mathcal{A}_A + [\mathcal{A}_A, \mathcal{A}_B\},\end{aligned} \tag{3.36}$$

where $R_{AB} = [\nabla_A, \nabla_B\} = \lambda^{\dot\alpha}\lambda^{\dot\beta}[\nabla_{A\dot\alpha}, \nabla_{B\dot\beta}\}$. Next one verifies that $\nabla_{[A}\mathcal{A}_{B\}}$ actually vanishes, which is due to Eqs. (3.3) and due to Bianchi identities of the curvature of ∇. Upon explicitly computing the commutator $[\mathcal{A}_A, \mathcal{A}_B\}$ and upon comparing it with R_{AB} thereby using Eqs. (2.49), one realizes that $R_{AB} = -[\mathcal{A}_A, \mathcal{A}_B\}$. In showing this, one also needs to use the property that $\mathscr{M}$ is right-flat. Therefore, we have verified the following fact:

Proposition 3.8. *The $\mathcal{F}^-$-part of the curvature $\mathcal{F}$ of the local supertwistor connection $\mathcal{D}$ on the local supertwistor bundle $\mathscr{T}$ over a civilized right-flat complex quaternionic RC supermanifold $\mathscr{M}$ is zero. Hence, the curvature $\mathcal{F}$ is self-dual, that is, $\mathcal{F} = \mathcal{F}^+$.*

Putting it differently, the connection $\mathcal{D}$ is flat on any β-surface $\pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathscr{M}$ for all $z \in \mathscr{P}$. This is going to be important in the paragraph subsequent to the following one, where we will show that the bundle of first-order jets of the dual universal line bundle $\mathscr{L}$ over the supertwistor space, i.e. $\mathrm{Jet}^1\mathscr{L}^{-1} \to \mathscr{P}$, corresponds to the dual of the bundle of local supertwistors $\mathscr{T}^\vee \to \mathscr{M}$ by means of the Penrose-Ward transform.

§3.13. Penrose-Ward transform. In this paragraph, we briefly discuss the general form of the Penrose-Ward transform which relates certain holomorphic vector bundles over the supertwistor space $\mathscr{P}$ to holomorphic vector bundles over $\mathscr{M}$ and vice versa. However, we merely quote the result. A detailed proof goes along the lines presented by Manin [51] and can be done in the supersymmetric setting without difficulties.

Suppose we are given a locally free sheaf $\mathscr{E}_\mathscr{P}$ on $\mathscr{P}$. Suppose further that $\mathscr{E}_\mathscr{P}$ is free when restricted to any submanifold $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{P}$ for all $x \in \mathscr{M}$. In addition, let $\Omega^1\mathscr{F}/\mathscr{P}$ the sheaf of relative differential one-forms on $\mathscr{F}$ as given by the sequence (3.12). Furthermore, let

$$\mathcal{D}_{T\mathscr{F}/\mathscr{P}} \,:\, \pi_2^*\mathscr{E}_\mathscr{P} \;\longrightarrow\; \pi_2^*\mathscr{E}_\mathscr{P} \otimes \pi_1^*\Omega^1\mathscr{M} \;\stackrel{\mathrm{id}\otimes\mathrm{res}}{\longrightarrow}\; \pi_2^*\mathscr{E}_\mathscr{P} \otimes \Omega^1\mathscr{F}/\mathscr{P} \tag{3.37}$$

be the relative connection on the pull-back $\pi_2^*\mathscr{E}_\mathscr{P}$ of $\mathscr{E}_\mathscr{P}$ to the correspondence space $\mathscr{F}$.

In order for the below theorem to work, one needs

$$\Omega^1\mathscr{M} \;\cong\; \pi_{1*}\Omega^1\mathscr{F}/\mathscr{P}, \tag{3.38}$$

since only then $\mathcal{D}_{T\mathscr{F}/\mathscr{P}}$ gives rise to a connection $\mathcal{D} := \pi_{1*}(\mathcal{D}_{T\mathscr{F}/\mathscr{P}})$ on $\mathscr{E}_\mathscr{M} = \pi_{1*}(\pi_2^*\mathscr{E}_\mathscr{P})$. One may check that this isomorphism indeed follows from the sequence (3.7) after dualizing and upon applying the direct image functor. In showing this, one uses the fact that the direct images $\pi_{1*}(\pi_2^*\mathscr{N}^\vee_{\mathbb{P}^1|\mathscr{P}})$ and $\pi^1_{1*}(\pi_2^*\mathscr{N}^\vee_{\mathbb{P}^1|\mathscr{P}})$ vanish due to Serre duality. Since the fibres of $\pi_1 \,:\, \mathscr{F} \to \mathscr{M}$ are compact and connected and the ones of $\pi_2 \,:\, \mathscr{F} \to \mathscr{P}$ are connected and simply connected (recall that $\mathscr{M}$ is assumed to be civilized), we have the following theorem:

Theorem 3.3. *Let $\mathscr{M}$ be a civilized right-flat complex quaternionic RC supermanifold and $\mathscr{P}$ its associated supertwistor space. Then there is a natural one-to-one correspondence between:*

- *(i) the category of locally free sheaves $\mathscr{E}_\mathscr{P}$ on $\mathscr{P}$ which are free on any submanifold $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{P}$ for all $x \in \mathscr{M}$ and*
- *(ii) the category of pairs $(\mathscr{E}_\mathscr{M}, \mathcal{D})$, where $\mathscr{E}_\mathscr{M}$ is a locally free sheaf on $\mathscr{M}$ given by $\mathscr{E}_\mathscr{M} = \pi_{1*}(\pi_2^*\mathscr{E}_\mathscr{P})$ and $\mathcal{D}$ is the push-forward of the relative connection on $\mathscr{F}$, i.e. $\mathcal{D} := \pi_{1*}(\mathcal{D}_{T\mathscr{F}/\mathscr{P}})$ which is flat on any β-surface $\pi_1(\pi_2^{-1}(z)) \hookrightarrow \mathscr{M}$ for all $z \in \mathscr{P}$.*

Notice that flatness on any β-surface is equivalent to saying that the curvature of $\mathcal{D}$ is self-dual. The above correspondence is called *Penrose-Ward transform*.

§3.14. Penrose-Ward transform of $\mathscr{T}$. Consider the bundle of local supertwistors $\mathscr{T}$ as defined in §3.11. As we have shown in §3.12., the local supertwistor connection is self-dual, i.e. flat on any β-surface in $\mathscr{M}$. So one naturally asks for the Penrose-Ward transform of $\mathscr{T}$. The answer gives the following proposition:

Proposition 3.9. *The Penrose-Ward transform takes the bundle of local supertwistors $\mathscr{T}$ over $\mathscr{M}$ to the dual of the sheaf of first-order jets $\mathrm{Jet}^1\mathscr{L}^{-1}$ of the dual universal line bundle $\mathscr{L}$ over $\mathscr{P}$.*

Proof: As a first check, notice that the restriction of $\mathscr{L}$ to any $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{P}$ is $\mathscr{O}_{\mathbb{P}^1}(-1)$. Hence, the dual restricts to $\mathscr{O}_{\mathbb{P}^1}(1)$. Hence, one may check that the dual of the sheaf of first-order jets $(\mathrm{Jet}^1\mathscr{L}^{-1})^\vee$ of $\mathscr{L}^{-1}$ is free when restricted to $\pi_2(\pi_1^{-1}(x)) \hookrightarrow \mathscr{P}$ as a consequence of the Euler sequence.[11] So, $(\mathrm{Jet}^1\mathscr{L}^{-1})^\vee$ satisfies point (i) of Thm. 3.3.

In the following, we are using a supersymmetric generalization of an argument by LeBrun [68]. Let $m_t : \mathscr{L} \setminus \{0\} \to \mathscr{L} \setminus \{0\}$ (zero section deleted), where $t \in \mathbb{C} \setminus \{0\}$, denote the scalar multiplication map. Furthermore, let $m_{t*} : T\mathscr{L} \to T\mathscr{L}$ be its Jacobian. According to LeBrun [68], one has an isomorphism

$$\mathrm{Jet}^1\mathscr{L}^{-1} \;\cong\; (\mathscr{L} \otimes (T\mathscr{L}/\{m_{t*}\}))^\vee.$$

Thus, we are about to verify that

$$\mathscr{T} \;\cong\; \pi_{1*}(\pi_2^*(\mathscr{L} \otimes (T\mathscr{L}/\{m_{t*}\}))).$$

To do this, we first recall that a point ℓ of $\mathscr{L}$ is a pair $(\pi_1(\pi_2^{-1}(z)), \lambda_{\dot\alpha})$, where $z \in \mathscr{P}$ and $\lambda_{\dot\alpha}$ is an auto-parallel tangent spinor, i.e. it satisfies Eq. (3.3). Therefore, a tangent vector at $\ell \in \mathscr{L}$ can be represented by Jacobi fields as introduced and discussed in the proof of Prop. 3.3. In particular, we may write

$$(\omega^A, \pi_{\dot\alpha}) \;=\; (J^{A\dot\beta}\lambda_{\dot\beta}, J^{B\dot\beta}\nabla_{B\dot\beta}\lambda_{\dot\alpha}),$$

for the tangent vector at $\ell \in \mathscr{L}$. From our dicussion given in §3.12., we know that such $(\omega^A, \pi_{\dot\alpha})$ satisfy

$$\begin{aligned} \lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\omega^B + \delta_A{}^B\pi_{\dot\alpha}) \;&=\; 0,\\ \lambda^{\dot\alpha}(\nabla_{A\dot\alpha}\pi_{\dot\beta} + (-)^{p_B}(R_{AB\dot\alpha\dot\beta} - \Lambda_{AB}\epsilon_{\dot\alpha\dot\beta})\omega^B) \;&=\; 0, \end{aligned}$$

i.e. they are annihilated by the local supertwistor connection (3.33). Since the transformation $m_t : \lambda_{\dot\alpha} \mapsto t\lambda_{\dot\alpha}$ induces

$$(\omega^A, \pi_{\dot\alpha}) \;\mapsto\; (t\omega^A, t\pi_{\dot\alpha}),$$

we conclude that the Penrose-Ward transform takes $\mathscr{T}$ to $\mathscr{L} \otimes (T\mathscr{L}/\{m_{t*}\})$, that is, to $(\mathrm{Jet}^1\mathscr{L}^{-1})^\vee$.

□

This leads us to the following interesting result:

[11] Recall that the Euler sequence is given by: $0 \longrightarrow \mathscr{O}_{\mathbb{P}^n} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(1) \otimes \mathbb{C}^{n+1} \longrightarrow T\mathbb{P}^n \longrightarrow 0$.

Proposition 3.10. *There are natural isomorphisms of Berezinian sheaves:*

(i) $\mathrm{Ber}\,\mathscr{J} \cong \mathscr{L}^{-1}$,
(ii) $\mathrm{Ber}(\mathscr{P}) := \mathrm{Ber}\,\Omega^1\mathscr{P} \cong \mathscr{L}^{4-\mathcal{N}}$.

Proof: Starting point is the jet sequence (3.23). This sequence in particular implies that

$$0 \longrightarrow \Omega^1\mathscr{P}\otimes\mathscr{L}^{-1} \longrightarrow \mathrm{Jet}^1\mathscr{L}^{-1} \longrightarrow \mathscr{L}^{-1} \longrightarrow 0,$$

i.e.

$$0 \longrightarrow \mathscr{J}^\vee \longrightarrow \mathrm{Jet}^1\mathscr{L}^{-1} \longrightarrow \mathscr{L}^{-1} \longrightarrow 0,$$

by virtue of Prop. 3.3. Therefore, we obtain a natural isomorphism of Berezinian sheaves[12]

$$\mathrm{Ber}\,\mathscr{J}\otimes\mathscr{L} \;\cong\; \mathrm{Ber}\,(\mathrm{Jet}^1\mathscr{L}^{-1})^\vee.$$

Since[13]

$$\pi_2^*((\mathrm{Jet}^1\mathscr{L}^{-1})^\vee) \;\cong\; \pi_1^*\mathscr{T}$$

and due to Eq. (3.28), we may conclude that

$$\mathrm{Ber}\,(\mathrm{Jet}^1\mathscr{L}^{-1})^\vee \;\cong\; \mathscr{O}_\mathscr{P},$$

which, in fact, proves point (i).

To verify point (ii), we merely apply Prop. 3.3. again. Indeed, from $T\mathscr{P}\cong\mathscr{J}\otimes\mathscr{L}^{-1}$ we find that

$$\mathrm{Ber}(\mathscr{P}) \;=\; \mathrm{Ber}\,\mathscr{J}^\vee\otimes\mathscr{L}^{3-\mathcal{N}} \;=\; (\mathrm{Ber}\,\mathscr{J}\otimes\mathscr{L})^\vee\otimes\mathscr{L}^{4-\mathcal{N}} \;\cong\; \mathscr{L}^{4-\mathcal{N}}.$$

This completes the proof.

□

3.4. Supertwistor space ($\mathcal{N}=4$)

Let us now discuss the $\mathcal{N}=4$ case. However, we can be rather brief on this, as the discussion is very similar to the one given above. Furthermore, for the sake of illustration we only discuss the hyper-Kähler case.

§3.15. Conic structure and β-plane bundle. Let $\mathscr{M}$ be a (4|8)-dimensional RC supermanifold equipped with the Levi-Civita connection. Recall again the sequence (2.9). It is equivalent to

$$0 \longrightarrow \mathscr{E}[1]\otimes\widetilde{\mathscr{S}}[-1] \longrightarrow T\mathscr{M} \longrightarrow \mathscr{S}[1]\otimes\widetilde{\mathscr{S}}[-1] \longrightarrow 0. \tag{3.39}$$

The reason for making this particular choice will become clear momentarily.

[12]Note that $\mathrm{Ber}\,\mathscr{L}\cong\mathscr{L}$.
[13]Recall that $\pi_2^*(\mathrm{Jet}^1\mathscr{L}^{-1})^\vee$ is free when restricted to $\pi_1^{-1}(x)$ for all $x\in\mathscr{M}$.

Let now $\mathscr{F}$ be the relative projective line bundle $P_{\mathscr{M}}(\widetilde{\mathscr{S}}^\vee[1])$ on $\mathscr{M}$. As before, the tangent bundle sequence induces a canonical (2|4)-conic structure on $\mathscr{M}$, which in local coordinates is given by

$$\begin{aligned} \mathscr{F} &\to G_{\mathscr{M}}(2|4; T\mathscr{M}), \\ [\lambda_{\dot\alpha}] &\mapsto \mathscr{D} := \langle \lambda^{\dot\alpha} E_{A\dot\alpha} \rangle. \end{aligned} \tag{3.40}$$

By a similar reasoning as given in Prop. 3.1., this distribution will be integrable if and only if $\mathscr{M}$ is right-flat. As before, $\mathscr{F}$ will be called the β-plane bundle in this case. In addition, Eq. (3.3) is then substituted by

$$\lambda^{\dot\alpha} D_{A\dot\alpha} \lambda_{\dot\beta} = 0, \tag{3.41}$$

i.e. $\lambda_{\dot\alpha}$ is auto-parallel with respect to the Levi-Civita connection on the β-plane $\Sigma \hookrightarrow \mathscr{M}$. Furthermore, as directly follows from the transformation laws given in Prop. 2.5., this equation *is* scale invariant since $\lambda_{\dot\alpha}$ is chosen to be a section of $\widetilde{\mathscr{S}}^\vee[1]$. This explains, why we have twisted $\widetilde{\mathscr{S}}$ by $\mathscr{O}_{\mathscr{M}}[k]$, with $k = -1$ in (3.39).

§3.16. Supertwistor space. As before, we obtain the following double fibration:

$$\begin{tikzcd} & \mathscr{F} \arrow[dl, "\pi_2"'] \arrow[dr, "\pi_1"] & \\ \mathscr{P} & & \mathscr{M} \end{tikzcd} \tag{3.42}$$

Here, $\mathscr{P}$ is the (3|4)-dimensional supertwistor space of $\mathscr{M}$. Again, we need to assume that $\mathscr{M}$ is civilized.

As already indicated, we shall now directly jump to the hyper-Kähler case. The following then gives the inverse construction.

Theorem 3.4. *There is a one-to-one correspondence between civilized complex hyper-Kähler supermanifolds $\mathscr{M}$ of dimension* (4|8) *and* (3|4)*-dimensional complex supermanifolds $\mathscr{P}$ such that:*

(i) $\mathscr{P}$ is a holomorphic fibre bundle $\pi : \mathscr{P} \to \mathbb{P}^1$ over $\mathbb{P}^1$,

(ii) $\mathscr{P}$ is equipped with a (4|8)*-parameter family of sections of π, each with normal bundle $\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ described by*

$$0 \longrightarrow \Pi\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^4 \longrightarrow \mathscr{N}_{\mathbb{P}^1|\mathscr{P}} \longrightarrow \mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \longrightarrow 0,$$

and

(iii) there exists a holomorphic relative symplectic structure ω of weight 2 *on $\mathscr{P}$.*

In proving this result, one basically follows the argumentation given in Sec. 3.1. The only modification is the replacement of $\widetilde{\mathscr{S}}$ by $\widetilde{\mathscr{S}}[-1] = \widetilde{\mathscr{S}} \otimes \mathrm{Ber}\,\widetilde{\mathscr{S}}$. In this respect, we also point out that triviality of the bundle $\widetilde{\mathscr{S}}[-1]$ certainly implies triviality of $\widetilde{\mathscr{S}}$.

§3.17. Remark. It is obvious, how to define the universal line bundle, the Jacobi bundle and the bundle of local supertwistors in the context of the $\mathcal{N} = 4$ supertwistor space. Prop. 3.3. can be modified accordingly. Point (ii) of Prop. 3.10. is then substituted by the fact that the Berezinian sheaf $\mathrm{Ber}(\mathscr{P})$ is globally trivial, i.e. $\mathrm{Ber}(\mathscr{P}) \cong \mathscr{O}_{\mathscr{P}}$. Hence, the $\mathcal{N} = 4$ supertwistor space is a formal Calabi-Yau supermanifold.

4. Real structures

So far, we have been discussing only complex supermanifolds. The subject of this section is to comment on a real version of theory.

§4.1. Almost quaternionic supermanifolds. Let us first present an overview about *real structures* on complex supermanifolds.

Definition 4.1. (Manin [51]) *A real structure on a complex supermanifold* $(\mathscr{M}, \mathscr{O}_{\mathscr{M}})$ *of type* $(\epsilon_1, \epsilon_2, \epsilon_3)$*, where* $\epsilon_i = \pm 1$ *for* $i = 1, 2, 3$*, is an even* $\mathbb{R}$*-linear mapping* $\rho : \mathscr{O}_{\mathscr{M}} \to \mathscr{O}_{\mathscr{M}}$ *such that*

$$\rho(\alpha f) \;=\; \bar{\alpha}\rho(f), \qquad \rho(\rho(f)) \;=\; (\epsilon_1)^{p_f} f, \qquad \rho(fg) \;=\; \epsilon_3(\epsilon_2)^{p_f p_g}\rho(g)\rho(f),$$

where f *and* g *are local holomorphic functions on* $\mathscr{M}$ *and* $\alpha \in \mathbb{C}$*. The bar means complex conjugation. Furthermore,* $\rho(f(\cdot)) = \overline{f(\rho(\cdot))}$*.*

If $\mathscr{E}$ *is a holomorphic vector bundle over* $\mathscr{M}$*, then a prolongation* $\hat{\rho}$ *of type* $\eta = \pm 1$ *of a given real structure* $\rho : \mathscr{O}_{\mathscr{M}} \to \mathscr{O}_{\mathscr{M}}$ *is an even* $\mathbb{R}$*-linear mapping* $\hat{\rho} : \mathscr{E} \to \mathscr{E}$ *such that*

$$\hat{\rho}(\hat{\rho}(\sigma)) \;=\; \eta(\epsilon_1)^{p_\sigma}\sigma, \qquad \hat{\rho}(f\sigma) \;=\; \epsilon_3(\epsilon_2)^{p_f p_\sigma}\hat{\rho}(\sigma)\rho(f), \qquad \hat{\rho}(\sigma f) \;=\; \epsilon_3(\epsilon_2)^{p_f p_\sigma}\rho(f)\hat{\rho}(\sigma),$$

where σ *is a local section of* $\mathscr{E}$ *and* f *is a local holomorphic function on* $\mathscr{M}$*. If* $\eta = +1$ *then the prolongation is called real while for* $\eta = -1$ *quaternionic.*

Having recalled the definition of real structures and their extensions to vector bundles, we may now give the following definition:

Definition 4.2. *A* $(4|2\mathcal{N})$*-dimensional RC supermanifold* $\mathscr{M}$ *is called an almost quaternionic RC supermanifold if there is a real structure* ρ *on* $\mathscr{M}$ *of type* $(-1, 1, 1)$ *which leaves* $\mathscr{E} \otimes \widetilde{\mathscr{S}}$ *invariant and which induces two quaternionic prolongations* $\hat{\rho}_1 : \mathscr{S} \to \mathscr{S}$ *and* $\hat{\rho}_2 : \widetilde{\mathscr{S}} \to \widetilde{\mathscr{S}}$*, respectively. In addition, it is also assumed that* ρ *has a (real)* $(4|2\mathcal{N})$*-dimensional supermanifold* $\mathscr{M}_\rho$ *of* ρ*-stable points in* $\mathscr{M}$*.*

§4.2. Structure group on $\mathscr{M}_\rho$. In §2.4., we have discussed the form of the structure group G of $T\mathscr{M}$. If $\mathscr{M}$ is equipped with an almost quaternionic structure, G may be reduced to the real form G_ρ on $\mathscr{M}_\rho$ which is described by[14]

$$1 \;\longrightarrow\; \mathbb{Z}_{|4-\mathcal{N}|} \;\longrightarrow\; S(GL(1|\tfrac{1}{2}\mathcal{N}, \mathbb{H}) \times GL(1|0, \mathbb{H})) \;\longrightarrow\; G_\rho \;\longrightarrow\; 1, \tag{4.1}$$

where $G_\rho \subset GL(4|2\mathcal{N}, \mathbb{R})$. This makes it clear why it is necessary to have an *even* number $\mathcal{N}$ of supersymmetries as otherwise one cannot endow an RC supermanifold with an almost quaternionic structure.

Furthermore, a scale is defined in this case as follows:

[14]See Salamon [70] for the purely bosonic situation.

Definition 4.3. *A scale on an almost quaternionic RC supermanifold $\mathscr{M}$ is a choice of a particular non-vanishing volume form $\tilde{\varepsilon} \in H^0(\mathscr{M}, \mathrm{Ber}\,\widetilde{\mathscr{S}}^\vee)$ on the vector bundle $\widetilde{\mathscr{S}}$ such that the corresponding volume form* $\mathrm{Vol} \in H^0(\mathscr{M}, \mathrm{Ber}(\mathscr{M}))$ *obeys* $\rho(\mathrm{Vol}) = \mathrm{Vol}$.

Clearly, a choice of scale reduces the structure group G_ρ on $\mathscr{M}_\rho$ further down to SG_ρ, which in fact is given by

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow SU(2|\mathcal{N}) \times SU(2|0) \longrightarrow SG_\rho \longrightarrow 1, \tag{4.2}$$

as follows from (2.18).

Now one can basically repeat the analysis given in Secs. 2.2. and 2.3. starting from almost quaternionic RC supermanifolds. One eventually arrives at the notions of quaternionic, quaternionic Kähler and hyper-Kähler structures, that is, in Defs. 2.3., 2.4. and 2.5. one simply needs to remove the word "complex".

§4.3. Supertwistor space. It remains to clarify the additional structure on the supertwistor space $\mathscr{P}$ needed in order to be associated with an RC supermanifold equipped with a real structure in the above sense.

On first notices that by starting from $\mathscr{M}$, the real structure ρ on $\mathscr{M}$ naturally induces real structures on $\mathscr{F}$ and $\mathscr{P}$, respectively, which are, of course, of the same type as ρ, that is, $(-1,1,1)$. For instance, since ρ is assumed to have a quaternionic prolongation $\hat{\rho}_2 : \widetilde{\mathscr{S}} \to \widetilde{\mathscr{S}}$, the induced real structure acts on the fibres of $\pi_1 : \mathscr{F} \to \mathscr{M}$ as the antipodal map $(\lambda_{\dot{1}}, \lambda_{\dot{2}}) \mapsto (-\bar{\lambda}_{\dot{2}}, \bar{\lambda}_{\dot{1}})$. Since $\mathscr{P}$ foliates $\mathscr{F}$, one obtains the induced real structure on $\mathscr{P}$. The following theorem clarifies also the reverse direction:

Theorem 4.1. *There is a one-to-one correspondence between:*

(i) *civilized right-flat quaternionic RC supermanifolds $\mathscr{M}$ of (complex) dimension* $(4|2\mathcal{N})$ *and*

(ii) $(3|\mathcal{N})$*-dimensional complex supermanifolds $\mathscr{P}$ each containing a family of holomorphically embedded projective lines $\mathbb{P}^1$ each having normal bundle $\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ inside $\mathscr{P}$ described by* (3.6) *and in addition, $\mathscr{P}$ has a real structure of type* $(-1,1,1)$ *which is compatible with the above data and which acts on the projective lines $\mathbb{P}^1$ as the antipodal map.*

Proof: In fact, almost everything has been proven (cf. also Thm. 3.1.). It remains to show that by going from (ii) $\to$ (i) the antipodal map on $\mathscr{P}$ indeed gives the correct real structure on $\mathscr{M}$. To see that the induced real structure ρ on $\mathscr{M}$ yields two quaternionic prolongations $\hat{\rho}_1 : \mathscr{S} \to \mathscr{S}$ and $\hat{\rho}_2 : \widetilde{\mathscr{S}} \to \widetilde{\mathscr{S}}$, respectively, we apply arguments of Hitchin et al. [30].

In particular, consider $\widetilde{\mathscr{S}} = \pi_{1*}(\pi_2^*\mathscr{O}_{\mathbb{P}^1}(1))$. Then the prolongation $\hat{\rho}_2$, induced by the antipodal map on $\mathbb{P}^1$, is given by

$$\hat{\rho}_2(a_{\dot{\alpha}}\lambda^{\dot{\alpha}}) \ := \ \bar{a}_{\dot{2}}\lambda^{\dot{1}} - \bar{a}_{\dot{1}}\lambda^{\dot{2}}.$$

Analogously, the antipodal map induces a quaternionic prolongation $\hat{\rho}_1$ on the bundle $\mathscr{S} = \pi_{1*}(\pi_2^*(\mathscr{O}_{\mathbb{P}^1} \otimes \mathbb{C}^2))$.

$\square$

In a similar fashion, one may make the appropriate changes in Thm. 3.2.

Finally, we have the following fact:

Proposition 4.1. *Let $\mathscr{M}$ be a civilized right-flat quaternionic RC supermanifold and $\mathscr{P}$ its associated supertwistor space. Then there is a natural diffeomorphism $\mathscr{P} \cong P_{\mathscr{M}_\rho}(\widetilde{\mathscr{S}^\vee}|_{\mathscr{M}_\rho})$. Hence, we obtain a nonholomorphic fibration*

$$\mathscr{P} \;\to\; \mathscr{M}_\rho$$

of the supertwistor space over $\mathscr{M}_\rho \subset \mathscr{M}$. Typical fibres of this fibration are two-spheres S^2.

§4.4. Remark. In the purely bosonic setting and for Euclidean signature, the twistor space has an alternative definition which is equivalent to the definition in terms of the projectivization of the right-chiral spin bundle. Let M be an oriented Riemannian four-manifold. The twistor space P of M can equivalently be defined as the associated bundle (Atiyah et al. [72])

$$P \;:=\; P(M, SO(4)) \times_{SO(4)} (SO(4)/U(2)) \tag{4.3}$$

with

$$P \;\to\; M. \tag{4.4}$$

Typical fibres of this bundle are two-spheres $S^2 \cong SO(4)/U(2)$ which parametrize almost complex structures on the fibre T_xM of TM over $x \in M$. Recall that an almost complex structure $\mathcal{J}$ is an endomorphism of the tangent bundle that squares to minus the identity, i.e. $\mathcal{J}^2 = -1$. Note that while a manifold M admits in general no almost complex structure, its twistor space P can always be equipped with an almost complex structure $\mathcal{J}$ (Atiyah et al. [72]). Furthermore, $\mathcal{J}$ is integrable if and only if the Weyl tensor of M is self-dual [71, 72]. Then P is a complex three-manifold with an antiholomorphic involution ρ which maps $\mathcal{J}$ to $-\mathcal{J}$ and the fibres of the bundle (4.4) over $x \in M$ are ρ-invariant projective lines $\mathbb{P}^1$, each of which has normal bundle $\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2$ in the complex manifold P. Here and in the following we make no notational distinction between real structures appearing on different (super)manifolds.

In the supersymmetric setting, the situation is slightly different. Let us consider $\mathscr{M}_\rho$ from above. The tangent spaces $T_x\mathscr{M}_\rho$ for $x \in \mathscr{M}_\rho$ are isomorphic to $\mathbb{R}^{4|2\mathcal{N}}$. So almost complex structures are parametrized by the supercoset space[15] $OSp(4|2\mathcal{N})/U(2|\mathcal{N})$, which is a supermanifold of (real) dimension $2 + \mathcal{N}(\mathcal{N}+1)|4\mathcal{N}$, and whose even part is[16]

$$(SO(4) \times Sp(2\mathcal{N}, \mathbb{R}))/(U(2) \times U(\mathcal{N})) \;\cong\; SO(4)/U(2) \times Sp(2\mathcal{N}, \mathbb{R})/U(\mathcal{N}). \tag{4.5}$$

[15]For more details, see e.g. Wolf [12].

[16]Recall that if G is a Lie supergroup and H a closed Lie subsupergroup (i.e. $H_{\rm red}$ is closed in $G_{\rm red}$) then $G/H := (G_{\rm red}/H_{\rm red}, \mathscr{O}_{G/H})$, where $\mathscr{O}_{G/H}(\mathscr{U}) := \{f \in \mathscr{O}_G(\pi^{-1}(\mathscr{U}))\,|\,\tilde{\phi}f = {\rm pr}^*f\}$ with $\mathscr{U} \subset H_{\rm red}$, $\pi : G_{\rm red} \to G_{\rm red}/H_{\rm red}$ and ${\rm pr}\colon G \times H \to G$ are the canonical projections and $\varphi = (\phi, \tilde{\phi}) : G \times H \to G$ is the right action of H on G. See Kostant [73] for more details. Hence, $(G/H)_{\rm red} \equiv G_{\rm red}/H_{\rm red}$.

Thus, the supertwistor space $\mathscr{P} \to \mathscr{M}_\rho$, as viewed as in Prop. 4.1., cannot be reinterpreted as a space which does describe all possible almost complex structures on $\mathscr{M}_\rho$. Nevertheless, one can view $\mathscr{P}$ as a space describing a certain class of almost complex structures on $\mathscr{M}_\rho$. Remember that the complexified tangent bundle $T\mathscr{M}_\rho \otimes \mathbb{C}$ can be factorized as $T\mathscr{M}_\rho \otimes \mathbb{C} \cong \mathscr{H} \otimes \widetilde{\mathscr{S}}$. In particular, these complex structures, being compatible with this tangent bundle structure, are again parametrized by two-spheres $S^2 \cong SO(4)/U(2)$ and are given in a structure frame by (cf. Wolf [12])

$$\mathbf{J}_{A\dot{\alpha}}{}^{B\dot{\beta}} \;=\; -\mathrm{i}\delta_A{}^B \frac{\lambda_{\dot{\alpha}}\hat{\lambda}^{\dot{\beta}} + \lambda^{\dot{\beta}}\hat{\lambda}_{\dot{\alpha}}}{\lambda_{\dot{\gamma}}\hat{\lambda}^{\dot{\gamma}}}, \tag{4.6}$$

where $\lambda_{\dot{\alpha}}$ are homogeneous coordinates on $\mathbb{P}^1$ ($\cong S^2$) and ${}^t(\hat{\lambda}^{\dot{\alpha}}) := (\bar{\lambda}^{\dot{2}}, -\bar{\lambda}^{\dot{1}})$ (see also the preceding paragraph). Now one may introduce an almost complex structure $\mathscr{J}$ on $\mathscr{P} \xrightarrow{S^2} \mathscr{M}_\rho$ by setting $\mathscr{J}_z = \mathbf{J}_z \oplus J_z$ for $z \in \mathscr{P}$. Here, $\mathbf{J}_z$ is given in terms of (4.6) and J_z in terms of the standard almost complex structure on S^2, respectively. In fact, following the arguments of Atiyah et al. [72], this description of $\mathscr{J}$ does not depend on the choice of local coordinates. Hence, $\mathscr{J}_z$ can be defined for all $z \in \mathscr{P}$ and thus, $\mathscr{P}$ comes equipped with a natural almost complex structure.

Next one can show that this almost complex structure is integrable if and only if $\mathscr{M}$ is right-flat and furthermore that the fibres of $\mathscr{P} \to \mathscr{M}_\rho$ are ρ-invariant projective lines $\mathbb{P}^1$ each having normal bundle $\mathscr{N}_{\mathbb{P}^1|\mathscr{P}}$ inside $\mathscr{P}$ described by

$$0 \;\longrightarrow\; \Pi\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^{\mathcal{N}} \;\longrightarrow\; \mathscr{N}_{\mathbb{P}^1|\mathscr{P}} \;\longrightarrow\; \mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathbb{C}^2 \;\longrightarrow\; 0. \tag{4.7}$$

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Appendix

A Proof of Prop. 2.6.

Subject of this appendix is to give a proof of Prop. 2.6. First, we show the second relation of Eqs. (2.49). The proof of the third one follows similar lines as for the second one. So we omit it at this point and leave it to the reader. Eventually, we prove the first relation.

The curvature components $R_{ABC}{}^D$ can be decomposed into irreducible[17] pieces as

$$\begin{aligned} R_{ABC}{}^D &= C_{ABC}{}^D + D_{ABC}{}^D + E_{AB}\delta_C{}^D + \\ &\quad + (\mathcal{N}-2)(-)^{p_C(p_A+p_B)} E_{C\{A}\delta_{B]}{}^D - 2(-)^{p_C(p_A+p_B)}\Lambda_{C\{A}\delta_{B]}{}^D, \end{aligned} \tag{A.1}$$

where $C_{ABC}{}^D$ and Λ_{AB} obey the properties stated in Prop. 2.6. and $D_{ABC}{}^D = D_{\{AB]C}{}^D$, $D_{\{ABC]}{}^D = 0$ and $E_{AB} = E_{\{AB]}$. Furthermore, $D_{ABC}{}^D$ is totally trace-free.

Recall the Bianchi identity

$$R_{[A\dot\alpha B\dot\beta C\dot\gamma\}}{}^{D\dot\delta} = 0,$$

which reads explicitly as

$$R_{A\dot\alpha B\dot\beta C\dot\gamma}{}^{D\dot\delta} + (-)^{p_A(p_B+p_C)} R_{B\dot\beta C\dot\gamma A\dot\alpha}{}^{D\dot\delta} + (-)^{p_C(p_A+p_B)} R_{C\dot\gamma A\dot\alpha B\dot\beta}{}^{D\dot\delta} = 0. \tag{A.2}$$

Upon substituting

$$R_{A\dot\alpha B\dot\beta C\dot\gamma}{}^{D\dot\delta} = \big[\epsilon_{\dot\alpha\dot\beta} R_{ABC}{}^D + R_{A(\dot\alpha B\dot\beta)C}{}^D\big]\delta_{\dot\gamma}{}^{\dot\delta} + \big[\epsilon_{\dot\alpha\dot\beta} R_{AB\dot\gamma}{}^{\dot\delta} + R_{A(\dot\alpha B\dot\beta)\dot\gamma}{}^{\dot\delta}\big]\delta_C{}^D, \tag{A.3}$$

which follows from (2.48) and upon contracting with $\epsilon_{\dot\delta\dot\epsilon}$, we arrive at

$$\begin{aligned} &\big[\epsilon_{\dot\alpha\dot\beta} R_{ABC}{}^D + R_{A(\dot\alpha B\dot\beta)C}{}^D\big]\epsilon_{\dot\gamma\dot\delta} + \big[\epsilon_{\dot\alpha\dot\beta} R_{AB\dot\gamma\dot\delta} + R_{A(\dot\alpha B\dot\beta)\dot\gamma\dot\delta}\big]\delta_C{}^D + \\ &\quad + (-)^{p_A(p_B+p_C)}\big[\epsilon_{\dot\beta\dot\gamma} R_{BCA}{}^D + R_{B(\dot\beta C\dot\gamma)A}{}^D\big]\epsilon_{\dot\alpha\dot\delta} + \big[\epsilon_{\dot\beta\dot\gamma} R_{BC\dot\alpha\dot\delta} + R_{B(\dot\beta C\dot\gamma)\dot\alpha\dot\delta}\big]\delta_A{}^D + \\ &\quad\quad + (-)^{p_C(p_A+p_B)}\big[\epsilon_{\dot\gamma\dot\alpha} R_{CAB}{}^D + R_{C(\dot\gamma A\dot\alpha)B}{}^D\big]\epsilon_{\dot\beta\dot\delta} + \big[\epsilon_{\dot\gamma\dot\alpha} R_{CA\dot\beta\dot\delta} + R_{C(\dot\gamma A\dot\alpha)\dot\beta\dot\delta}\big]\delta_B{}^D = 0. \end{aligned} \tag{A.4}$$

Therefore, upon looking at the terms proportional to $\epsilon_{\dot\alpha\dot\beta}\epsilon_{\dot\gamma\dot\delta}$ (plus a permutation of the indices), one arrives after some lengthy but straightforward calculations at

$$(-)^C R_{ABC}{}^C = 0 \qquad \text{and} \qquad (-)^C R_{\{ABC]}{}^C = 0. \tag{A.5}$$

In addition,

$$\begin{aligned} R_{ABC}{}^D &= \tfrac{1}{3}(R_{ABC}{}^D + (-)^{p_Bp_C} R_{ACB}{}^D + (-)^{p_A(p_B+p_C)} R_{BCA}{}^D) + \\ &\quad + \tfrac{1}{3}(R_{ABC}{}^D - (-)^{p_Bp_C} R_{ACB}{}^D) + \\ &\quad\quad + \tfrac{1}{3}(R_{ABC}{}^D - (-)^{p_A(p_B+p_C)} R_{BCA}{}^D) \\ &= R_{\{ABC]}{}^D + \tfrac{2}{3} R_{A[BC\}}{}^D + \tfrac{2}{3} R_{B[AC\}}{}^D. \end{aligned} \tag{A.6}$$

By comparing this result with Eqs. (A.1) and (A.5), we conclude that $D_{ABC}{}^D$ and E_{AB} must vanish and $R_{\{ABC]}{}^D = C_{ABC}{}^D$. Hence,

$$R_{ABC}{}^D = C_{ABC}{}^D - 2(-)^{p_C(p_A+p_B)}\Lambda_{C\{A}\delta_{B]}{}^D, \tag{A.7}$$

which is the desired result.

[17]Note that in order to obtain the set of independent superfield components, one has to go one step further and employ the second Bianchi identity (see e.g. Eq. (2.53)).

Let us now discuss the first relation given in Eqs. (2.49). By looking at terms in Eq. (A.4) which are symmetric in $\dot\alpha,\dot\beta$ but antisymmetric in $\dot\gamma,\dot\delta$, we find that

$$\begin{aligned}&\left[R_{A(\dot\alpha B\dot\beta)C}{}^D+2(-)^{p_C(p_A+p_B)}R_{C[A\dot\alpha\dot\beta}\delta_{B\}}{}^D\right]+\\&\quad+\ \left[R_{A(\dot\alpha B\dot\beta)C}{}^D+(-)^{p_A(p_B+p_C)}R_{B(\dot\alpha C\dot\beta)A}{}^D+(-)^{p_C(p_A+p_B)}R_{C(\dot\alpha A\dot\beta)B}{}^D\right]\ =\ 0.\end{aligned}\tag{A.8}$$

However, the second line vanishes identically as it represents a Bianchi identity for the curvature of the bundle $\mathscr{H}\to\mathscr{M}$. Therefore, we end up with

$$R_{A(\dot\alpha B\dot\beta)C}{}^D\ =\ -2(-)^{p_C(p_A+p_B)}R_{C[A\dot\alpha\dot\beta}\delta_{B\}}{}^D.\tag{A.9}$$

Finally, we notice that the form (2.50) of the Ricci tensor can straightforwardly be obtained by substituting Eqs. (2.49) into its definition and by explicitly performing the appropriate index traces. This remark concludes the proof of Prop. 2.6.

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